A Proof by Mathematical Induction

We will use mathematical induction to prove that: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n+1) \).

The proof has two parts:

I. A proof that the formula works for the first case, \( i=1 \).
II. A proof that if the formula works for the \( k \)th case, it works for the \( (k+1) \)st case.

Part I.

Substitute \( n = 1 \) into the formula: \( \sum_{i=1}^{1} i = 1 = \frac{1}{2}(1)(1+1) \). The formula works for this case.

Part II.

Assume that the formula works for the \( k \)th case: \( \sum_{i=1}^{k} i = \frac{1}{2}k(k+1) \).

Prove that the formula works for the \( (k+1) \)st case: \( \sum_{i=1}^{k+1} i = \frac{1}{2}(k+1)(k+2) \).

If we show that \( \sum_{i=1}^{k} i + (k+1) = \sum_{i=1}^{k+1} i \), this will complete the proof.

Does \( \frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}(k+1)(k+2) \)? A bit of algebra shows that this is correct.

Therefore, we have proven that \( \sum_{i=1}^{n} i = \frac{1}{2}n(n+1) \).

Exercise 1: Prove by mathematical induction that \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \).

Exercise 2: Using the formulas given above for \( \sum_{i=1}^{n} i \) and \( \sum_{i=1}^{n} i^2 \), obtain a formula for the sum of the squares of the first \( n \) consecutive odd numbers. That is find an expression in terms of \( n \) for \( \sum_{i=1}^{n} (2i-1)^2 \).
Moment of inertia of a long, uniform cylinder

Problem: A thin cylindrical rod of length $d$ has uniform density. The rod is rotated at one end about an axis transverse to its length. Find the rotational inertia of the rod about this axis.

Starting the solution: Position the $x$ axis along the longitudinal axis of the rod as in the diagram to the right. Let the origin be on the axis of rotation. Divide the rod into $n$ equal elements of width $\Delta x$ such that $\Delta x = d/n$. If the entire mass of the rod is denoted as $M$, then the mass of element $\Delta x$ is $\Delta m = M/n$. Finally, let the position of the center of mass of the $i$th element be $x_i$, where the index $i$ goes from 1 to $n$.

Now let’s calculate the rotational inertia for different values of $n$. We will be using the approximate formula:

$$I_n = \sum_{i=1}^{n} \frac{M}{n} x_i^2$$

The plan is make $n$ larger and larger and eventually take the limit of $I_n$ as $n$ approaches infinity. For $n = 1$, $x_i = d/2$, and the rotational inertia in this crude approximation is:

$$I_1 = \sum_{i=1}^{1} \frac{M}{1} x_i^2$$

$$= M (d/2)^2$$

$$= M d^2 / 4$$

For $n = 2$, the values of $x_i$ for the two elements are $d/4$ and $3d/4$. Then we have:

$$I_2 = \frac{M}{2} [(d/4)^2 + (3d/4)^2]$$

$$= (5/16) Md^2$$

For $n = 4$, the values of $x_i$ are $d/8, 3d/8, 5d/8, 7d/8$. We see a pattern of odd numbers emerging. Expressing $I_4$ in such a way as to show this pattern explicitly, we have:
Completing the solution: Generalize the result obtained for $I_4$ to any value of $n$. That means you’ll need to replace the denominators 4 and 64 with functions of $n$. You’ll also need to write a summation expression in terms of the index, $i$, for the sum of the squares of the first $n$ odd integers. The induction proof that you did earlier should be useful. Once you have an expression for $I_n$ evaluate it in the limit that $n$ goes to infinity and $\Delta x$ becomes infinitesimal. Your final result should be $I = fMd^2$, where $f$ is a fraction that you are to determine.