Problems with Vectors, Calculus, and 2-Dimensional Motion

1. Consider the parametric equations for horizontal and vertical displacements of a projectile:

\[ x = x_o + v_{ox} t \quad \text{and} \quad y = y_o + v_{oy} t + a_y t^2 / 2. \]

a. Eliminate the variable, t, to obtain the trajectory of the projectile in the form, \( y = f(x) \).
b. Sketch a graph of the trajectory.
c. Evaluate \( dy/dx \).
d. Set your result for \( dy/dx \) equal to zero and solve for the \( x \)-coordinate at maximum height. Express your result in simplest form in terms of \( x_o, v_{ox}, v_{oy}, \) and \( a_y \). Why does this method work?
e. Determine the \( y \)-coordinate at the point of maximum height in terms of \( y_o, v_{oy}, \) and \( a_y \). Express your answer in simplest form.
f. Now examine your results. Do they make sense? Are the units right? the signs? Do the results agree with physics you already know?

For the next problem, you’ll need the following relationships:

\[ \frac{d (\sin \theta)}{dt} = \cos \theta \quad \text{and} \quad \frac{d (\cos \theta)}{dt} = -\sin \theta \]

You’ll also need to know how to compute derivatives of the form \( \frac{d [f(\theta)]}{dt} \). Note that \( f \) is a function of \( \theta \) and that \( \theta \) is a function of \( t \), the variable with respect to which the derivative is taken. The derivative is computed as follows using the chain rule:

\[ \frac{d [f(\theta)]}{dt} = \frac{df}{d\theta} \frac{d\theta}{dt} \]

2. A particle is moving counterclockwise in a circular path of radius, \( r \), at constant angular velocity, \( \omega \), as shown in the diagram.

a. Write the position vector \( \vec{r} \) of the particle when it is at point \( P \) in terms of \( r \) \((= |\vec{r}|)\), \( \theta \), and unit vectors.
b. Find the velocity vector of the particle by taking the derivative of the position vector with respect to time. This will require taking derivatives of the form \( \frac{d (\sin \theta)}{dt} \), where \( \theta \) is a function of \( t \). The result in this case is \( \frac{d (\sin \theta)}{dt} = \cos \theta \frac{d \theta}{dt} \). Note that \( \frac{d \theta}{dt} \) is simply \( \omega \), the angular velocity.
c. Prove that the velocity vector is perpendicular to the position vector.
d. Find the acceleration vector of the particle by taking the derivative of the velocity vector with respect to time.
e. Show that the acceleration vector is opposite in direction to the position vector. That is, the acceleration vector points radially inward.
f. Determine the magnitude of the acceleration vector.
g. Check your results. Make sure they’re consistent with what you already know about circular motion at constant speed.