Analyzing an LCR circuit

1. Begin by studying the handout, “Finishing the Damped Harmonic Oscillator Problem.” (hereafter termed the DHO handout) Then do this: Determine expressions for the constants \( B \) and \( D \) in terms of the constants \( A \) and \( \phi \).

Now we’ll look at an analogous problem. A circuit is composed of an inductor, \( L \), and a capacitor, \( C \). The charge, \( q \), on the capacitor oscillates at the angular frequency, \( \omega = \sqrt{1/LC} \). When a resistor, \( R \), is placed in series with the inductor and capacitor, the angular frequency of oscillation is given by \( \omega' = \sqrt{\omega^2 - \delta^2} \), where \( \delta = R/2L \). Use the given symbols in showing your work for the problems below.

2. Write the differential equation that describes the oscillation of the LCR circuit in terms of the variables \( q \) and \( t \) and the constants \( L \), \( C \), \( R \). Use the symbols \( \dot{q} \) and \( \ddot{q} \) to denote the time derivatives of \( q \).

3. Solve the differential equation to obtain a solution similar in form to the solution obtained for the damped harmonic oscillator. Note that you are solving for \( q(t) \) rather than \( x(t) \). Show the steps in your solution as done in class and on the DHO handout.

In addition to the constants given previously, let \( q_o \) represent the initial charge on the capacitor, and let \( D \) represent a quantity analogous to \( D \) in the DHO handout.

4. Suppose that the current in the circuit is 0 at \( t = 0 \). Find an expression for \( D \) in terms of \( \delta \) and \( \omega' \).

5. For the given initial conditions, determine \( \dot{q}(t) \), the current as a function of time. Express your result in simplest form in terms of \( \delta, \omega, \omega' \), and \( q_o \) (but not \( D \)). Reduce your equation to a single term. Check your equation to make sure that it gives the expected result for the undamped case.

6. For the given initial conditions, determine \( \ddot{q}(t) \), the rate of change of current as a function of time. Check your equation to make sure that it gives the expected result for the undamped case.

7. Write an expression for the total energy stored in the capacitor and the inductor at any time, \( t \). Then take the time derivative and simplify to give the time rate of change of the total energy in terms of \( \dot{q}(t) \) and \( R \). Tell why your result makes sense. (This problem can be done without substituting in the solutions for \( q, \dot{q} \) or \( \ddot{q} \). Keep it simple!)

8. Given these values: \( L = 100 \) mH, \( C = 1000 \) µF, \( R = 10 \) Ω. At \( t = 0 \), the voltage across the capacitor is 1000 V. Construct a spreadsheet in which you plot \( \dot{q}(t) \) and \( \dot{E}(t) \) on the same time axis. All values must be in SI units. The numerical values of \( \dot{E}(t) \) will be much larger than \( \dot{q}(t) \). If you divide all the values of \( \dot{E}(t) \) by 500, the features of both functions will be clearly visible. Select the time scale to show at least 3 zero’s of the functions.

9. Why should the maxima (or minima) and the zero’s of the two functions plotted in problem 8 coincide?
10. Evaluate $|\dot{E}(t)|$ analytically. Then determine 2 times when the rate of energy loss is a maximum and 2 times when it is a minimum. Check that your results agree with the graph.

11. Explain, in terms of the physical properties of the system, why your results in problem 10 make sense.

12. It can be shown that the energy lost in the first cycle is the following fraction of the initial energy:

$$\frac{\Delta E}{E} = e^{-2\pi/Q} - 1,$$

where $Q = \omega L / R$.

For the given circuit, how much of the original energy remains after the first cycle?

Q is termed the quality factor. For a high-Q circuit, the fractional energy loss per cycle is low and is given approximately by $2\pi/Q$. Is the given circuit high or low Q?

13. Using your spreadsheet, design an LCR circuit that has a Q-value of 100 and oscillates at an angular frequency of 100 s$^{-1}$ (both values to 3 significant figures).

a. State the values of $L$, $C$, and $R$ that you selected.

b. Calculate the fractional energy loss per cycle.

c. When a generator whose frequency may be varied around the resonant frequency drives an LCR circuit, the width of the resonance peak as a fraction of the resonant frequency at one-third of the maximum current amplitude is given by $3^{\frac{1}{2}}/Q$. Call this the relative width of the peak. A high-Q circuit will have a narrow resonance peak. What is the relative width for your circuit?

d. In the spreadsheet, alter the scale of the graph as needed to plot the data for the circuit you designed. Note that you’ll no longer need the divisor of 500 for $dE/dt$. Be sure to change it in all the relevant cells.

e. Submit your spreadsheet file to BlackBoard using the filename lcr_xyz.xls