Distance Formulas

Solutions:

1. What is the closest the line $4x+5y=20$ comes to the origin?

   Since $4x+5y-20=0$, we have $d=\frac{|4(0)+5(0)-20|}{\sqrt{4^2+5^2}}=\frac{20}{\sqrt{41}}=\frac{20\sqrt{41}}{41}$.

2. What is the distance between the points $(2, f(2))$ and $\left(f(2),2\right)$ if $f(x)=\sqrt{4x+1}$.

   The two points are $(2,3)$ and $(3,2)$, so the distance is $\sqrt{(2-3)^2+(3-2)^2}=\sqrt{2}$.

3. How far from the origin are the points of intersection of the conics $x^2+7y^2=47$ and $2x^2-4y^2=28$?

   Since the distance we want is simply $\sqrt{x^2+y^2}$, notice that the linear combination using 1 times each equation is simply $3x^2+3y^2=75 \Leftrightarrow x^2+y^2=25$, so the points of intersection, whatever they are, are only 5 units from the origin.

4. What is the shortest distance between the circle $x^2+y^2=25$ and the line $3x+4y=48$?

   First notice that the closest the line gets to the origin is $\frac{|3(0)+4(0)-48|}{\sqrt{3^2+4^2}}=\frac{48}{5}$, so it is more than 5 units from the origin. Now subtract the radius of the circle, yielding the distance $\frac{48}{5}-5=\frac{23}{5}$.

5. Find the distance between the parallel lines $2x-3y=12$ and $2x-3y=36$.

   Pick any point on one line and plug it into the distance formula for a point and a line. So let's pick (6,0) from the first line. Now $d=\frac{|2(6)-3(0)-36|}{\sqrt{2^2+3^2}}=\frac{24}{\sqrt{13}}=\frac{24\sqrt{13}}{13}$.

6. A triangle has vertices (2,3), (6,-2), and (-1,-4). Find the area of the triangle.

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You could use the determinant-based formula to get the area (if you know it), or find the length of one segment and the altitude to it (using distance from a point to a line) and be done. If A = (2,3) and B = (6,-2), then $AB = \sqrt{4^2+5^2} = \sqrt{41}$. The equation for the line through A and B is $5x+4y-22=0$, so the altitude to AB has length $\frac{|5(-1)+4(-4)-22|}{\sqrt{5^2+4^2}} = \frac{43}{\sqrt{41}}$, making the area $\frac{1}{2}\left(\sqrt{41}\right)\left(\frac{43}{\sqrt{41}}\right) = \frac{43}{2}$.

7. Find the shortest distance between the parallel planes $3x+4y-5x=12$ and $3x+4y-5z=20$.

   Pick any point on the first plane, say (4,0,0) and plug this into $\frac{|3(4)+4(0)-5(0)-20|}{\sqrt{3^2+4^2+5^2}} = \frac{8}{\sqrt{50}} = \frac{8\sqrt{2}}{10} = \frac{4\sqrt{2}}{5}$.

8. What is the closest that the line $y = \frac{4}{7}x + \frac{1}{5}$ comes to a lattice point? Duke03-Team10.

   The line, in standard form, is $20x-35y+7=0$, so the distance to any lattice point (x,y) is $\frac{|20x-35y+7|}{\sqrt{20^2+35^2}}$, so the closest point will occur when the numerator is minimized. Since it is an absolute value, and must be an integer, we need to see if this quantity can be zero, or if not 0, what is the smallest possible value. This is a Diophantine problem. Well, clearly $20x-35y$ will always be a multiple of 5, so we could make it -5, making the entire numerator 2. This is the smallest possible value for the numerator, so the shortest distance is $\frac{2}{\sqrt{20^2+35^2}} = \frac{2}{5\sqrt{65}} = \frac{2\sqrt{65}}{325}$.

9. What is the closest the plane $z = \frac{1}{3}x + \frac{3}{5}y + \frac{4}{15}$ comes to a lattice point?

   Write the equation for the plane in standard form: $15x-5y-9z+4=0$. Now since 5 and 9 are relatively prime, the quantity $5y+9z$ can be made to equal any number we would like it to be (Chinese Remainder Theorem). When $x=0$, we can make $5y+9z=-4$, when $y=1, z=-1$ and the quantity $15x-5y-9z+4=0$ at the lattice point (0,1,-1), making the minimum distance zero.

10. Under what conditions would the line $Ax+By+C=0$ be guaranteed to hit a lattice point?

If A and B are relatively prime, then the linear combination $Ax + By = 1$ will always have a solution, so we can make $(Ax + By)(-C) = -C$, which would make the numerator of the distance formula $\dfrac{|Ax + By + C|}{\sqrt{A^2 + B^2}}$ equal to zero, hence the line hits some lattice point.

11. The circles $x^2 + y^2 + 10x - 6y + 18 = 0$ and $x^2 + y^2 - 14x + 4y + 44 = 0$ have two internal tangents. On each the distance between the points of tangency are equal. Find this distance.

11. $x^2 + y^2 + 10x - 6y + 18 = 0 \Leftrightarrow x^2 + 10x + 25 + y^2 - 6y + 9 = -18 + 25 + 9$

$\Leftrightarrow (x+5)^2 + (y-3)^2 = 16$

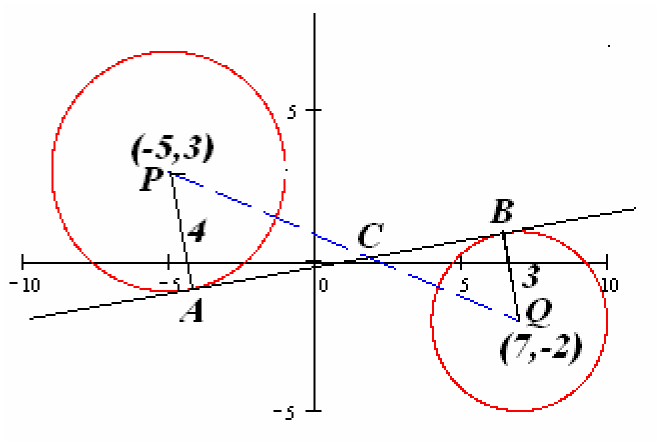
and

$x^2 + y^2 - 14x + 4y + 44 = 0 \Leftrightarrow x^2 - 14x + 49 + y^2 + 4y + 4 = -44 + 4 + 49$

$\Leftrightarrow (x-7)^2 + (y+2)^2 = 9$

The first circle has center (-5,3) and radius 4 while the second has center (7,-2) and radius 3. The distance between the centers P and Q is

$PQ = \sqrt{(7-(-5))^2 + (-2-3)^2} = \sqrt{12^2 + 5^2} = 13$



Since $\triangle APC \sim \triangle BQC$, we know that $\dfrac{AC}{BC} = \dfrac{PC}{QC} = \dfrac{4}{3}$. Thus $PC = \dfrac{4}{7}(13)$ and $QC = \dfrac{3}{7}(13)$. This makes $AC = \sqrt{\left(\dfrac{4 \cdot 13}{7}\right)^2 - 4^2} = \dfrac{4}{7}\sqrt{13^2 - 7^2} = \dfrac{4}{7}\sqrt{120} = \dfrac{8}{7}\sqrt{30}$.

Likewise $BC = \sqrt{\left(\dfrac{3 \cdot 13}{7}\right)^2 - 3^2} = \dfrac{3}{7}\sqrt{13^2 - 7^2} = \dfrac{3}{7}\sqrt{120} = \dfrac{6}{7}\sqrt{30}$. Thus the length of the internal tangent is $AB = \dfrac{8}{7}\sqrt{30} + \dfrac{6}{7}\sqrt{30} = 2\sqrt{30}$.