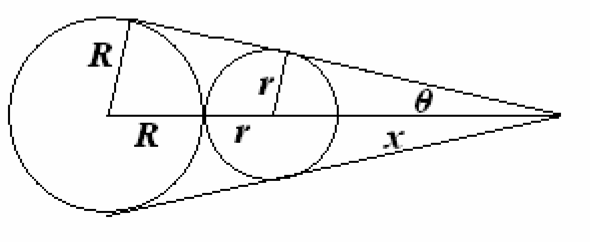
Geometry Facts – Circles & Cyclic Quadrilaterals

Solutions:

1. We want $\sin(2\theta)$.



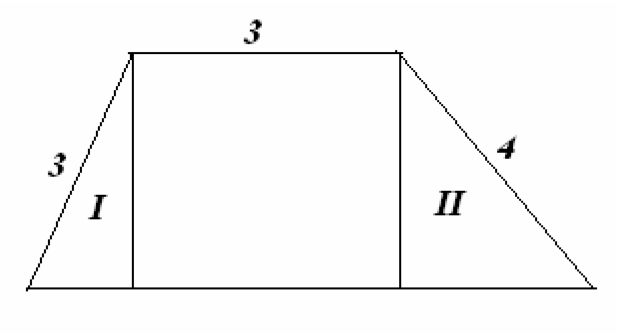
$$\frac{x}{x+(R=r)}=\frac{r}{R}\Rightarrow$$

$$x=\frac{r(R+r)}{R-r}\Rightarrow \sin\left(\theta\right)=\frac{r}{x}=\frac{R-r}{R+r}$$

$$\sin(2\theta)=2\sin(\theta)\cos(\theta)=2\left(\frac{R-r}{R+r}\right)\sqrt{1-\left(\frac{R-r}{R+r}\right)^2}=\frac{4(R-r)}{(R+r)^2}\sqrt{Rr}\ .$$

Of course, using numbers is easier. For R = 4, r = 1, $\sin(2\theta)={}^{24}\!/_{25}$. Since the angles can't be equal for these two soluti0ons, they must be complementary. Placing a 4" and a 9" circle in line with and tangent to opposite sides of a 1" circle, then drawing the appropriate external tangents, would produce a quadrilateral that is cyclic. Potential there for interesting puzzles?

2. Two angles must be supplementary, but they cannot be opposite. Thus the are consecutive, and the quadrilateral is a trapezoid, but not isosceles. Sliding triangles I and II together produces a 3-4-5 triangle , or area 6, with altitude to hypotenuse equal to 12/5 This makes the inner rectangle area equal to 2(1 / 5) = 36 / 5, for a total area of 6 + (36 / 5) = 66 / 5.



3. Remembering that Ptolemy's Theorem says that $ac+bd=pq$ (where a,b,c, and d are the lengths of the sides and p and q the diagonals), and using the inequalities

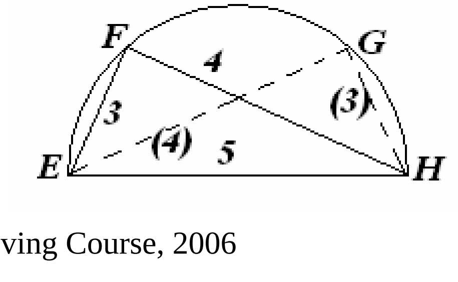
$$\left(ad+bc\right)^2\geq 0\Rightarrow (ad)^2+2(ad)(bc)+(bc)^2\geq 0\Rightarrow (ad)^2+(bc)^2\geq 2(ad)(bc)$$

$$\Rightarrow a^2d^2+b^2c^2\geq 2abcd\Rightarrow a^2d^2+\underline{b^2c^2}+b^2c^2+\underline{b^2d^2}\geq\underline{b^2c^2}+2abcd+\underline{b^2d^2}$$

$$\Rightarrow\left(a^2+b^2\right)\left(c^2+d^2\right)\geq\left(ac+bd\right)^2=(pq)^2.$$

Since all of the quantities are positive, we have $pq\leq\sqrt{\left(a^2+b^2\right)\left(c^2+d^2\right)}$

4. I. A. (7,24,25) and (15,20,25).



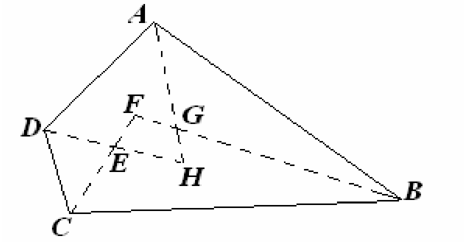
B. (35,120,125), (75,100,125) and (44,117,125). The first two come from the (7,24,25) and (3,4,5) triangles. A general approach would be to use the fact that $k(m^2-n^2), k(2mn),$ and $k(m^2+n^2)$, where *k*, m, n, are positive integers with m > n. produce all Pythagorean triplets. Setting $m^2+n^2=125$ leads to m = 10, n = 5, [producing (75,100,125)] or m = 11, n = 2 [producing (44,117,125); setting $m^2+n^2=25$, leads to m = 4, n = 3 [producing 7,24,25) for which we use k = 5].

II. A. Ptolemy's Theorem produces $FG(5)+(3)(3)=(4)(4) \Rightarrow FG = 7/5$. The answer is (17,7,15,25) [in any order].

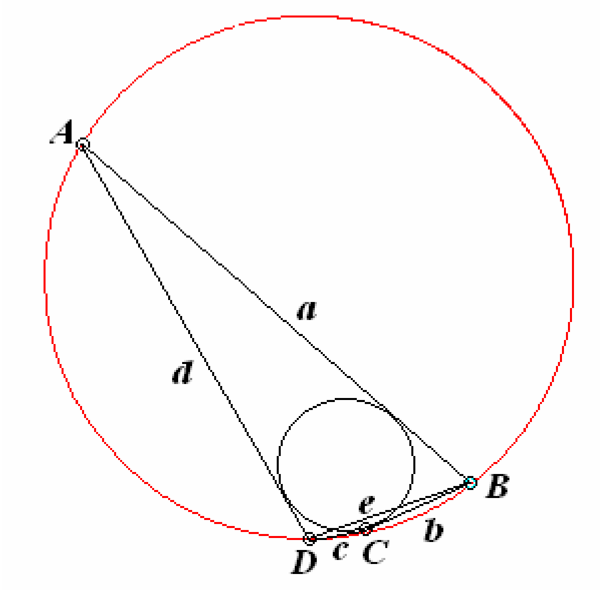
B. Using sides a,x,a,25 (each diagonal is $\sqrt{625-a^2}$) leads $x = 25 - 2a^2/25$. Positive integral x's come from a = 5, 19, or 15. The last leads to the solution in A. The other possibilities are (5,23,5,25) OR (10,17,10,25) [either answer in any order, is acceptable.]

C. Apply the theorem given in Part IA, for any appropriate d, to find n - 2 P3's of hypotenuse d^{n-2}. Build these triangles in a semicircle of diameter d^{n-2}. Connecting successive points on the semicircle produces a n-gon. Successive applications of Ptolemy's Theorem to find each side of the n-gon shows each must be rational. Multiplying all sides by the least common denominator involved produces a Pn.

5. The sum of angles CDE, DCE, BAG, and ABG is $\frac{1}{2}(360)=180^o$, so angles E and G must be supplementary. Thus quadrilateral EFGH is cyclic. Let $\measuredangle BAG = a, \measuredangle ABG = b,$ and $\measuredangle BCF = c.$ Then $a+b=180-G$ and $b+c=180-F$. Therefore $a-c=F-G=F-(180-E)=E+F-180$, so $A\_C = 2[E+F-180^o]=26^o$ In general, $A-C=2(E+F)-360$ and $B-D=2(E-F)$.



6. If we arrange the sides of the hexagon, keeping the same outer segments of the original circle attached, we must again produce a circle (of the same size)! Incidentally, the area of the hexagon will be preserved also. Arranging the sides in the order 2,7,11,2,7,11, shows that each triplet must fit in a semicircle. Applying Ptolemy's Theorem to the quadrilateral, (in a semicircle) of sides 2,7,11,D (for the diameter), we have $\left(\sqrt{D^2-121}\right)\left(\sqrt{D^2-4}\right)=7D+22$. This implies $D^3-174D-308=0,$ or $D^3-(2\cdot3\cdot29)D-(4\cdot7\cdot11)=0$. We note that $11<D<20$; also, if D is not irrational, it is integral. Hoping to find an integral D, we see that D would be divisible by 2 but not 3 or 4 (else 8 would divide the constant). The only possible integral D is then 14, which works.



7. Let the quadrilateral be ABCD. We will prove that angle A, chosen randomly, is "nice." Let AB = a, BC = b, CD = c and DA = d. Since ABCD is inscriptable, angles A and C are supplementary, so $\cos(C) = -\cos(A)$. Now

$$\cos(A) = \frac{a^2 + d^2 - e^2}{2ad}$$

$$= \frac{a^2 + d^2 - \left(b^2 + c^2 - 2bc\cos(A)\right)}{2ad}$$

Clearing fractions and solving for cos(A), we get $\cos(A) = \frac{\left(a^2 + d^2\right) - \left(b^2 + c^2\right)}{2\left(ad + bc\right)}$, which is rational, since the sides are integers. Since the quadrilateral can be circumscribed about a circle, it must be true that $a + c = b + d$. [This well-known fact can be proved by considering the lengths of the tangent segments to the circle from each vertex of ABCD.] This its semiperimeter s is equal to a+c (or b+d). Now Heros (actually Brahamgupta's) formula for the area of the inscribed quadrilateral is $K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$. Now we can replace s with either a+c or b+d, getting a wonderful formula for a quadrilateral that is both inscriptable and circumscriptable, $K = \sqrt{abcd}$. Since we are given that $abcd$ is a perfect square, our area is an integer. Finally noting that $\sin(C) = \sin(A)$, we see that the area K of ABCD is also given by $K = \frac{1}{2}\left(ad + bc\right)\sin(A)$, so $\sin(A) = \frac{2K}{ad + bc}$; since K is an integer, sin(A) is rational. Thus angle A is "nice", and our quadrilateral is "nice." Incidentally, the quadrilateral with the smallest perimeter with these properties has sides 1,2,9,8 (in that order). For those interested, it can be proven that the only regular polygon that is "nice" is the square.