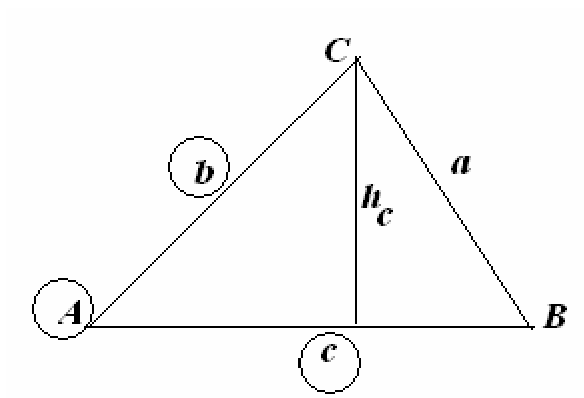
Triangles

Some of the following information is well known, but other bits are less known but useful, either in and of themselves (as theorems or formulas you might want to remember) or for the useful techniques and strategies used in their development.

Area Formulas:

Everyone knows the formula for the area of a triangle given a base and the corresponding height. It is, of course, $K = \frac{b \cdot h}{2}$. But if you are given 2 sides and the included angle (SAS), or all three sides (SSS), the following area formulas are useful.



In the above diagram, *A*, *b*, *c* are given, so we know that the altitude from *C* to side *c* has length $bSin(A)$, so the area is $K = \frac{(base)(height)}{2} = \frac{c \cdot bSin(A)}{2}$.

If, in the same diagram, only the sides are given, then we will use the Law of Cosines to find something about angle A. Note that we do not need to find the sin(A) from the cos(A), but will work with the square of the most recent formula.

$$a^2 = b^2 + c^2 - 2bd\cos(A) \Rightarrow \cos(A) = \frac{b^2 + c^2 - a^2}{2bc} \Rightarrow$$

$$\cos^2(A) = \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 \Rightarrow \sin^2(A) = 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2$$

$$K^2 = \frac{b^2c^2\sin^2(A)}{4} = \frac{b^2c^2\left(1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2\right)}{4}$$

$$= \frac{b^2c^2\left(\frac{(2bc)^2-(b^2+c^2-a^2)^2}{4b^2c^2}\right)}{4} = \frac{\left((2bc)^2-(b^2+c^2-a^2)^2\right)}{16}$$

$$= \frac{\left(2bc-(b^2+c^2-a^2)\right)\left(2bc+(b^2+c^2-a^2)\right)}{16}$$

$$= \frac{\left(a^2-(b-c)^2\right)\left((b+c)^2-a^2\right)}{16} = \frac{(a-(b-c))(a+(b-c))((b+c)-a)((b+c)+a)}{16}$$

Now if we let $P = a+b+c$ and $S = \frac{P}{2}$, we get

$$K^2 = \frac{(a+c-b)(a+b-c)(b+c-a)(a+b+c)}{16}$$

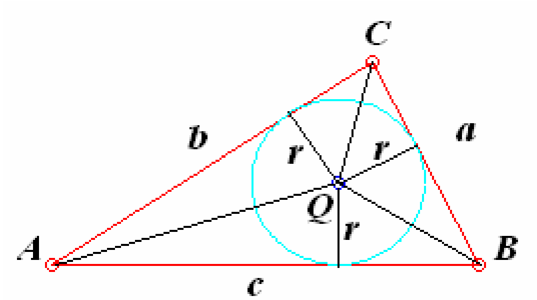
$$= \frac{P-2b}{2}\cdot\frac{P-2c}{2}\cdot\frac{P-2a}{2}\cdot\frac{P}{2} = (s-b)(s-c)(s-a)s$$

$$\therefore K^2 = s(s-a)(s-b)(s-c) \Rightarrow K = \sqrt{s(s-a)(s-b)(s-c)}$$

This last formula is known as Heron's Formula.

Special Points and Segments:

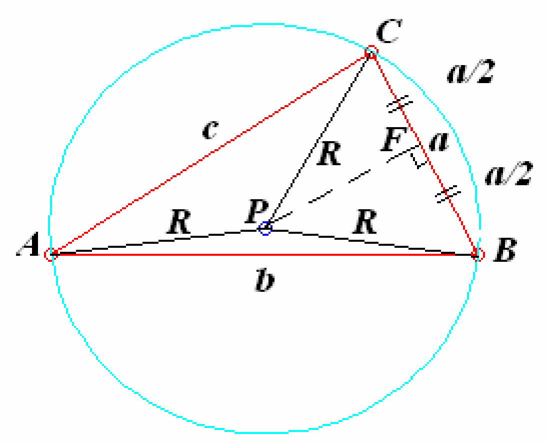
The angle-bisectors of the angles of a triangle are concurrent and meet at the point called the ***incenter***. Recall from your geometry that an angle-bisector is equidistant from the sides of the angle being bisected, so the incenter is equidistant from all three sides, making it the center of the circle that can be inscribed inside the triangle, the inscribed circle.



Since the area of the entire triangle is equal to the sum of the areas of the three smaller triangles $\triangle ABQ, \triangle BCQ, and \triangle ACQ$, we have

$$K = \frac{cr}{2} + \frac{ar}{2} + \frac{br}{2} = r\left(\frac{a+b+c}{2}\right) = rs \Rightarrow r = \frac{K}{s}$$

The perpendicular-bisectors of the sides meet at a point that is equidistant from the endpoints of each of the sides (the vertices), so this point is equidistant from all three vertices, making it the center of the circumscribed circle. This point is called the ***circumcenter***.

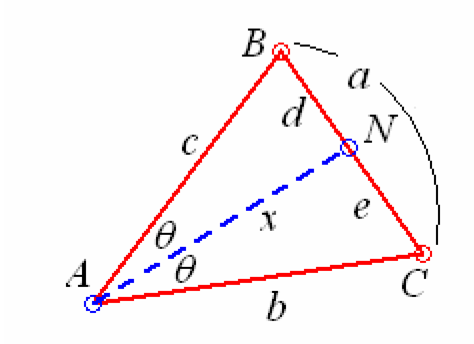


The area of the triangle, using the formula $K = \frac{1}{2}bc\sin(A)$ leads us to a formula that relates the area K to the radius R of the circumscribed circle. We know that $m\angle CPB = 2m\angle CAB$ (a central angle is measured by the arc and an inscribed angle is one-half the measure of the arc), so $m\angle FPB = m\angle A$, but in $\triangle FPB, \sin(\angle FPB) = \frac{a/2}{R}$, so $\sin(A) = \frac{a}{2R}$, making the area $K = \frac{1}{2}bc\sin(A) = \frac{1}{2}bc\left(\frac{a}{2R}\right) = \frac{abc}{4R} \Rightarrow R = \frac{abc}{4K}$.

Cevians:

A cevian is a line segment that joins a vertex of a triangle with a point on the opposite side. Medians and angle bisectors, and in some cases altitudes, are examples of cevians.

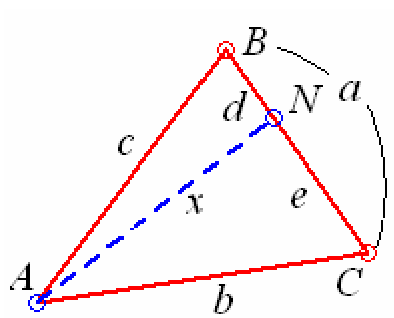
There are some nice formulas for the lengths of these segments. But first, let's review some of our basic geometry.



Theorem: An angle-bisector divides the opposite side into segments that are proportional to their adjacent sides. $\frac{c}{b}=\frac{d}{e}$, or $\frac{d}{c}=\frac{e}{b}$.

This means that if the length of segment ***BC*** has length *a*, then $d=\frac{ac}{b+c}$ and $e=\frac{bc}{b+c}$.

Stewart's Theorem.



Let's look at the proof of the general theorem for finding the length of a cevian. The Law of Cosines does the trick.

In $\triangle ABC$ $\cos(C)=\frac{a^2+b^2-c^2}{2ab}$ and in $\triangle ABN$ $Cos(C)=\frac{b^2+e^2-x^2}{2be}$, so

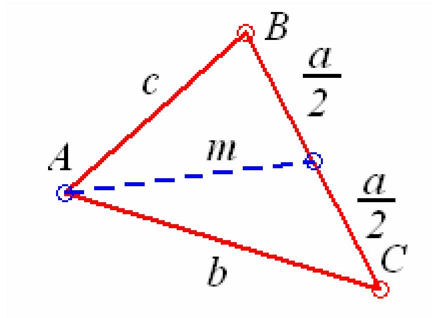
$$\frac{a^2+b^2-c^2}{2ab}=\frac{b^2+e^2-x^2}{2be}\Rightarrow e\left(a^2+b^2-c^2\right)=a\left(b^2+e^2-x^2\right)$$

$$\therefore ax^2+a^2e-ae^2=b^2\left(a-e\right)+c^2e$$

$$\Leftrightarrow ax^2+ae(a-e)=b^2d+c^2e$$

$$\Leftrightarrow ax^2+aed=b^2d+c^2e\Leftrightarrow a\left(x^2+de\right)=b^2d+c^2e$$

Now let's apply this to the median.



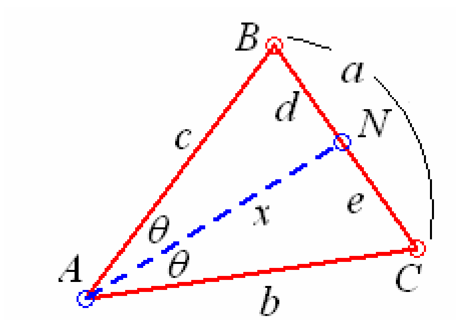
Using Stewart's Theorem we have

$$a\left(m^2+\left(\frac{a}{2}\right)^2\right)=b^2\left(\frac{a}{2}\right)+c^2\left(\frac{a}{2}\right)=\left(\frac{a}{2}\right)\left(b^2+c^2\right)$$

$$\therefore m^2+\left(\frac{a}{2}\right)^2=\frac{b^2+c^2}{2}\Rightarrow m^2=\frac{b^2+c^2}{2}-\frac{a^2}{4}$$

$$\therefore m=\sqrt{\frac{b^2+c^2}{2}-\frac{a^2}{4}}$$

If we apply Stewart's Theorem to the angle-bisector, we have



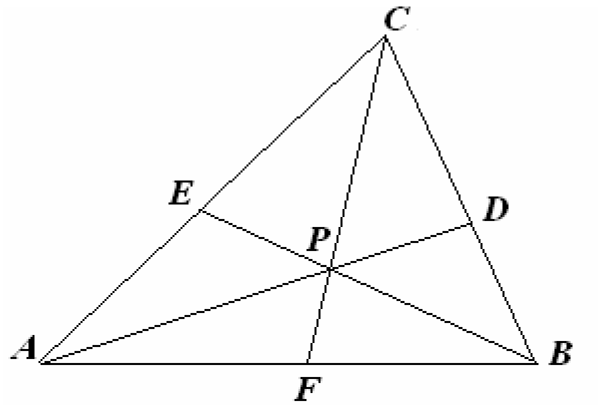
$$a\left(x^2+de\right)=db^2+ec^2\Rightarrow x^2=\frac{db^2+ec^2}{a}-de=\frac{\frac{ac}{b+c}b^2+\frac{ab}{b+c}c^2}{a}-\left(\frac{ac}{b+c}\right)\left(\frac{ab}{b+c}\right)$$

$$\Rightarrow x^2=\frac{b^2c+bc^2}{b+c}-\frac{a^2bc}{(b+c)^2}=\frac{bc\left((b+c)^2-a^2\right)}{(b+c)^2}\Rightarrow x=\sqrt{bc\left(1-\frac{a^2}{(b+c)^2}\right)}$$

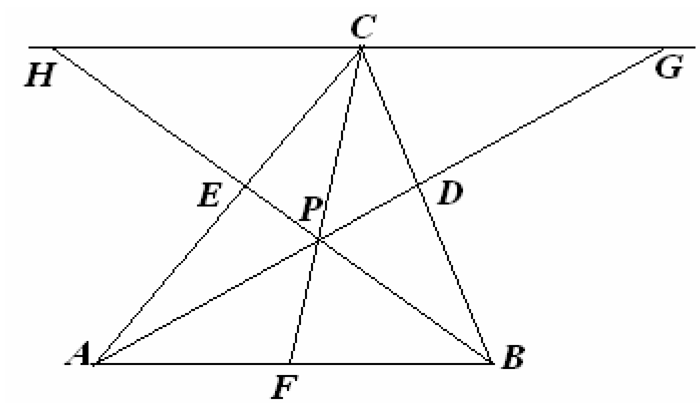
Ceva's Theorem:

Any time that the cevians of a triangle are concurrent, there is a special relationship that exists for the lengths of the segments on each of the sides. This relationship is known as

Ceva's Theorem and states that if cevians AD, BE and CF are concurrent at point P if and only if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$



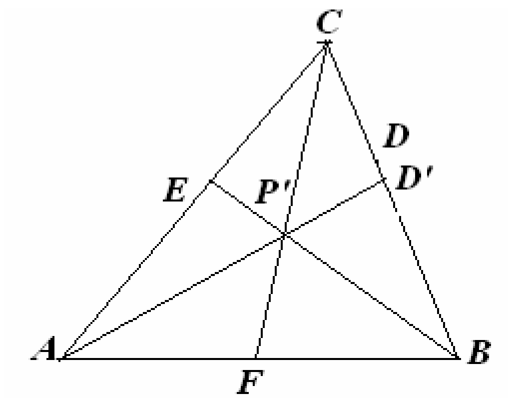
First extend segments AD and BE and construct line the line L through C that is parallel to AB. Label the points where AD and BE extended meet L as G and H.



From the following pairs of similar triangles we have the following proportions:

$\triangle AEB \sim \triangle CEH$	$\frac{AE}{CE} = \frac{AB}{CH}$ (1)
$\triangle ADB \sim \triangle GDC$	$\frac{CD}{BD} = \frac{CG}{AB}$ (2)
$\triangle FPB \sim \triangle CPH$	$\frac{FB}{CH} = \frac{FP}{CP}$ (3)
$\triangle APF \sim \triangle GPC$	$\frac{AF}{CG} = \frac{FP}{CP}$ (4)

From the last two proportions it follows that $\frac{FB}{CH}=\frac{FA}{CG}$ or $\frac{FB}{FA}=\frac{CH}{CG}$ (5). Now multiply (a), (2) and (5) together to get $\frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA}=\frac{AB}{CH}\cdot\frac{CG}{AB}\cdot\frac{CH}{CG}=1$. This proves that if the lines are concurrent, the identity is true. Now we must prove that if the identity is true then the cevians must be concurrent.



Let P' be ther point where FC and BE intersect and draw segment AP' and extend to point D' on side BC. Since these three cevians are concurrent, the first part of the theorem holds so

$$\frac{AE}{EC}\cdot\frac{CD'}{D'B}\cdot\frac{BF}{FA}=1,$$

but from the hypothesis we know that

$$\frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA}=1,$$

so

$$\frac{AE}{EC}\cdot\frac{CD'}{D'B}\cdot\frac{BF}{FA}=\frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA}\Rightarrow\frac{CD'}{D'B}=\frac{CD}{DB}$$

The last part of this proportion, $\frac{CD'}{D'B}=\frac{CD}{DB}$ implies that

$$\frac{CD'}{D'B}+1=\frac{CD}{DB}+1\Rightarrow\frac{CD'+D'B}{D'B}=\frac{CD+DB}{DB}\Rightarrow\frac{CB}{D'B}=\frac{CB}{DB}.$$

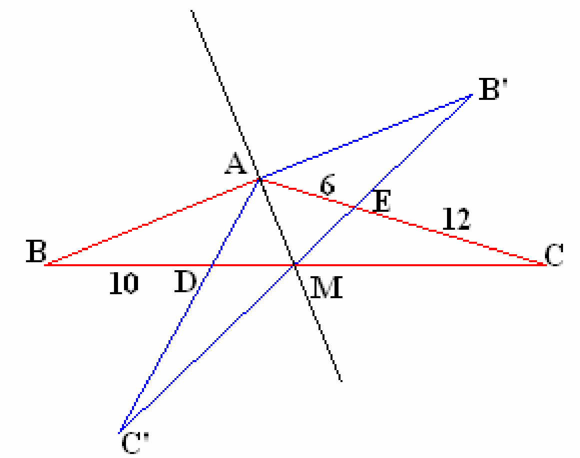
This last proportion implies that D and D' are the same point and the second part of the theorem is proved.

Problems:

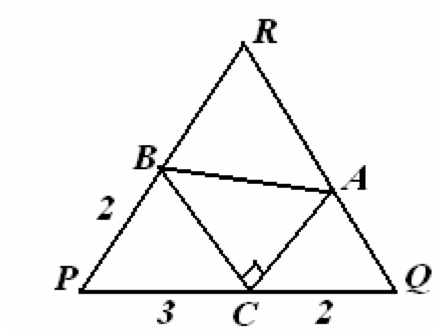
1. Two sides of a triangle are 7 and 9 while the median to the third side has length 5. Find the length of the third side.

2. The sides of a triangle are consecutive integers and the area is an integer. Find the triangle with the smallest perimeter that is not a right triangle. Are there others?

3. Triangle ABC is reflected in (or about) its median $\overline{AM}$ (extended) as shown. If $AE = 6,\ EC = 12,\ BD = 10$ and $AB = k\sqrt{3}$, compute *k*. **ARML 1978 Ind 8**



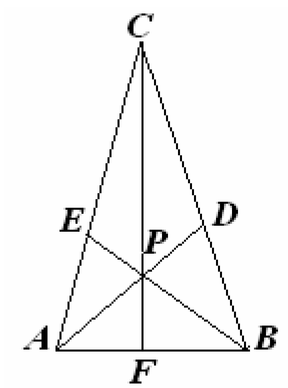
4. Right triangle ABC (hypotenuse $\overline{AB}$) is inscribed in equilateral triangle PQR, as shown. If $PC = 3$, and $BP = CQ = 2$, compute AQ. **ARML 1991 Ind 7**



5. In triangle ABC, the perpendicular bisector of $\overline{AC}$ intersects $\overline{AC}$ at M and $\overline{AB}$ at T. If the area of triangle AMT is $\frac{1}{4}$ the area of triangle ABC, and $\sphericalangle A + \sphericalangle C = 128^{\circ}$, compute the number of degrees in angle A.
ARML 1988 Ind 4

6. An isosceles triangle has a median equal to 15 and an altitude equal to 24. This information determines exactly two triangles. Compute the area of each of these triangles. **ARML 1994 Team 4**

7. Point P is inside $\triangle ABC$. Line segments $\overline{APD}$, $\overline{BPE}$, and $\overline{CPE}$ are drawn with D on $\overline{BC}$, E on $\overline{CA}$, and F on $\overline{AB}$. (See figure). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$. **AIME 1989 #15**



8. The points (0,0), (a,11), and (b,37) are the vertices of an equilateral triangle. Find the value of ab. **AIME 1994 # 8**

9. Two medians in a triangle are congruent if and only if the sides which they bisect are congruent.