Part I: Multiple Choice (Answers)

1. **Solution:** Let $f(x) = ax^2 + bx + c$.

$$\begin{cases} 9a + 3b + c = 15 \\ 9a - 3b + c = 9 \end{cases}$$
$$6b = 6$$
$$b = 1$$

Answer: 1

2. **Solution:** $16^{(x^2+3x-1)} = 8^{(x^2+3x+2)}$

$$2^{4(x^2+3x-1)} = 2^{3(x^2+3x+2)}$$
$$2^{4x^2+12x-4} = 2^{3x^2+9x+6}$$
$$4x^2 + 12x - 4 = 3x^2 + 9x + 6$$
$$x^2 + 3x - 10 = 0$$
$$(x+5)(x-2) = 0$$
$$x = -5 \text{ or } x = 2$$

Answer: -3

3. **Solution:** $(\log_{27} x^3)^2 = \log_{27} x^6$

$$(3\log_{27} x)^2 = 6\log_{27} x$$
$$3\log_{27} x(3\log_{27} x - 2) = 0$$
$$3\log_{27} x = 0 \quad \text{or} \quad \log_{27} x = \tfrac{2}{3}$$

Thus $\log_{27} x = 0$ or $x = 1$

$$\log_{27} x = \tfrac{2}{3} \text{ or } x = 27^{\frac{2}{3}} \text{ or } 9$$

Answer: 10

4. **Solution:** $|x| - x + y = 10$ and $x + |y| + y = 12$

Case I
$x > 0, y > 0$
$y = 10$
$x + 2y = 12$
$x + 20 = 12$
DOESN'T WORK

Case II
$x > 0, y < 0$
$y = 10$
$x = 12$
DOESN'T WORK

Case III
$x < 0, y < 0$
$-2x + y = 10$
$x = 12$
$-24 + y = 10$
$y = 34$
DOESN'T WORK

Case IV
$x < 0, y > 0$
$-2x + y = 10 \qquad y = 10 + 2x$
$x + 2y = 12$
$x = 12 - 2(10 + 2x)$
$x = 12 - 20 - 4x$
$5x = -8$
$x = -\tfrac{8}{5} \quad y = 10 + 2(-\tfrac{8}{5}) = \tfrac{34}{5}$
$x + y = -\tfrac{8}{5} + \tfrac{34}{5}$
$x + y = \tfrac{26}{5}$

Answer: $\tfrac{26}{5}$

5. **Solution:** $x+\frac{1}{x}=k$.

$$x^2+2+\frac{1}{x^2}=k^2$$

$$x^2+\frac{1}{x^2}=k^2-2$$

$$\left(x^2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)=\left(k^2-2\right)(k)$$

$$x^3+\frac{1}{x}+x+\frac{1}{x^3}=k^3-2k$$

$$\left(x^3+\frac{1}{x^3}\right)+\left(x+\frac{1}{x}\right)=k^3-2k$$

$$x^3+\frac{1}{x^3}+k=k^3-2k$$

$$x^3+\frac{1}{x^3}=k^3-3k$$

$$\left(x^3+\frac{1}{x^3}\right)\left(x^2+\frac{1}{x^2}\right)=\left(k^3-3k\right)\left(k^2-2\right)$$

$$x^5+\frac{1}{x}+x+\frac{1}{x^5}=k^5-3k^3-2k^3+6k$$

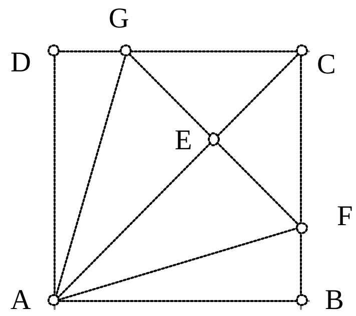
$$x^5+\frac{1}{x^5}+k=k^5-5k^3+6k$$

$$x^5+\frac{1}{x^5}=k^5-5k^3+5k$$

Sum of the coefficients $k^5-5k^3+5k=1-5+5=1$

Answer: 1

6. **Solution:** Draw diagonal AC. Let x represent the length of the side of the square.



$$x=10\cos 15^\circ$$

$$x^2=100\cos^2 15^\circ$$

$$2\cos^2 15^\circ=\cos 30^\circ+1$$

$$x^2=100\cos^2 15^\circ=50\left(\frac{\sqrt{3}}{2}+1\right)=25(\sqrt{3}+2)$$

Answer: $25(2+\sqrt{3})$

7. **Solution:** $\log_{10} 17^{10,000} = 12{,}304.49\ldots$ There are 12,305 digits

Answer: 12,305

8. **Solution:**

$$\tan A + \tan B + \tan C = k$$
$$\left[\frac{\tan A + \tan B}{1 - \tan A \tan B}\right][1 - \tan A \tan B] + \tan C = k$$
$$\tan(A + B)[1 - \tan A \tan B] + \tan C = k$$
$$\tan(180^\circ - C)[1 - \tan A \tan B] + \tan C = k$$
$$-\tan C[1 - \tan A \tan B] + \tan C = k$$
$$\therefore \tan A \tan B \tan C = k$$

Answer: k

9. **Solution:** There are a total of 2^{100} subsets. The number of subsets containing 0, 1, 2, 3, …, 49 elements is exactly the same as those containing 100, 99, 98, 97, …, 49 elements. Since the number of subsets with 50 elements is ${}_{100}C_{50}$, the answer is

$$\frac{2^{100} - {}_{100}C_{50}}{2} + {}_{100}C_{50} = 2^{99} + \frac{{}_{100}C_{50}}{2}.$$

Answer: 6.843×10^{29}

10. **Solution:** Let x denote the time that Johnny and Frankie spent mowing together

$$\frac{x}{8} + \frac{x}{6} = \frac{1}{2}$$
$$6x + 8x = 24$$
$$x = \tfrac{24}{14} \text{ or } \tfrac{12}{7}$$

Johnny must spend an additional 4 hours moving the other half so he spends

$$4 + \tfrac{12}{7} = \tfrac{40}{7} = 5\tfrac{5}{7} \text{ hrs.}$$

Answer: $5\tfrac{5}{7}$ hrs.

11. **Solution:** Let K denote the average on the 1^{st} $n - 2$ tests.

$$\frac{[K(n-2) + 2(100)]}{n} = 88$$
$$\frac{[K(n-2) + 2(28)]}{n} = 70$$
$$K(n-2) = 88n - 200$$
$$K(n-2) = 70n - 56$$
$$88n - 200 = 70n - 56$$
$$18n = 144$$
$$n = 8 \quad \text{Substituting we find } K = 84$$

If x is the lowest score Johnny can make on the final to have an 80 average, then

$$6(84) + 2x = 8(80) \quad \text{or} \quad x = 68$$

Answer: 68

12. **Solution:**

$$\begin{cases} x\sec q + y\tan q = 2\cos q \\ x\tan q + y\sec q = \cot q \end{cases}$$

By Cramer's Rule;

$$y = \frac{\begin{vmatrix} \sec\theta & 2\cos\theta \\ \tan\theta & \cot\theta \end{vmatrix}}{\begin{vmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{vmatrix}} = \frac{\sec\theta\cot\theta - 2\tan\theta\cos\theta}{\sec^2\theta - \tan^2\theta} = \frac{\frac{1}{\sin\theta} - 2\sin\theta}{1} = \frac{1 - 2\sin^2\theta}{\sin\theta} = \frac{\cos 2\theta}{\sin\theta}$$

Answer: $\frac{\cos 2q}{\sin q}$

13. **Solution:** Consider

a	b	c	d

If $a + b + c = 18$

$b + c + d = 18$

$a - d = 0$ or $a = d$, so pattern repeats. Therefore x must be 7, 8, or 3.

The only one which fits is 3.

Answer: 3

14. **Solution:** $532_b = 4(148)_b$

$5b^2 + 3b + 2 = 4(b^2 + 4b + 8) = 4b^2 + 16b + 32$

$b^2 - 13b - 30 = 0$ or $(b - 15)(b + 2) = 0$

$b = 15$

Answer: 15

15. **Solution:** $a * b = a + b + 1$

$a * (-1) = a + (-1) + 1 = a$ $\therefore$ identity element is -1

Calculate $(2 * 3)^{-1} * 2^{-1} * 3^{-1}$

$2 * 3 = 2 + 3 + 1 = 6$

$6^{-1} * 2^{-1} * 3^{-1}$ $\quad 2 + a + 1 = -1 \Rightarrow a = -4 \Rightarrow 2^{-1} = -4$

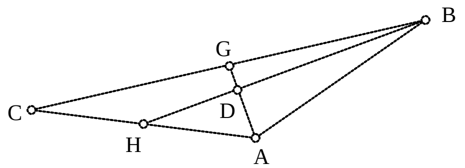
$(-8 * -4) * -5$ $\quad 6 + a + 1 = -1 \Rightarrow a = -8 \Rightarrow 6^{-1} = -8$

$-11 * -5$ $\quad 3 + a + 1 = -1 \Rightarrow a = -5 \Rightarrow 3^{-1} = -5$

$-11 + (-5) + 1 = -15$

Answer: -15

16. **Solution:** Since the medians intersect at a point twice as far from the vertex as the side, label the triangle as shown.



$\overline{AH} = \overline{HC} = 9$ and $\overline{CG} = \overline{GB} = 12$

$\overline{AD} = 2x$, $\overline{DG} = x$, $\overline{DB} = 2y$, and $\overline{DH} = y$

$(\overline{AB})^2 = (2x)^2 + (2y)^2 = 4(x^2 + y^2)$

$(2x)^2 + y^2 = 9^2$ and $x^2 + (2y)^2 = 12^2$

$5x^2 + 5y^2 = 81 + 144$

$x^2 + y^2 = 45$

$(\overline{AB})^2 = 4 \cdot 45$

Answer: $6\sqrt{5}$ $\qquad$ $\overline{AB} = 6\sqrt{5}$

17. **Solution:** Length of paper $= (400)(5.25) = 2100$

Average circumference $= \frac{23}{8}p$

Number of times wrapped around roll $= \dfrac{2100}{\frac{23}{8}p} = 232.5 \approx 233$

Answer: 233

18. **Solution:**

$x_1 < x_2 < x_3 < x_4 < x_5$

$\sum x = \dfrac{\sum \text{Sum}}{4} = \dfrac{9596}{4} = 2399$

$x_3 = 2399 - 375 - 1713 = 311$

$x_1 + x_3 = 424 \quad \Rightarrow \quad x_1 = 113$

$x_1 + x_2 = 375 \quad \Rightarrow \quad x_2 = 262$

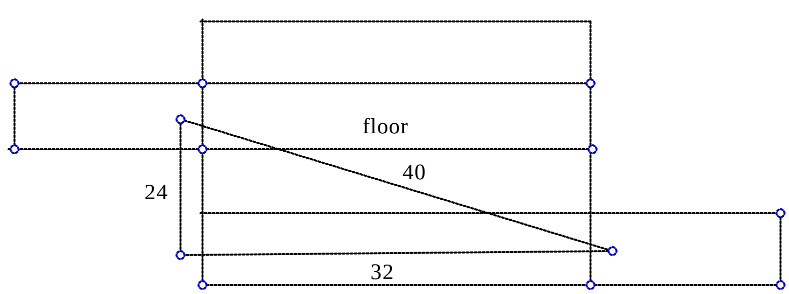
$(113)(262)(311) = 9207466$

Answer: 9207466 (none of these)

19. **Solution:** The solid is an octahedron each of whose faces is an equilateral triangle of side length $\sqrt{2}$ and area $\dfrac{\sqrt{3}}{2}$. Thus the surface area is $8 \cdot \dfrac{\sqrt{3}}{2} = 4\sqrt{3}$.

Answer: $4\sqrt{3}$

20. **Solution:**



Answer: 40

Part II: Integer Problems (Answers)

1. **Solution:**

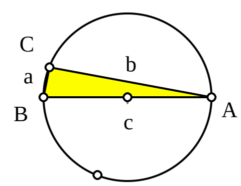
$$(x+iy)^3 = -74 + ki$$
$$x^3 + 3x^2yi - 3xy^2 - y^3i = -74 + ki$$
$$1 - 3y^2 = -74 \qquad 3y - y^3 = k$$
$$-3y^2 = -75 \qquad \pm 110 = k$$
$$y = \pm 5 \qquad \text{Take the absolute of } \pm 110\text{.}$$

Answer: 110

2. **Solution:** Note that each row ends with a perfect square. The row continuing 1000 will end with $32^2 = 1024$ and the next row will end in $33^2 = 1089$. The number below 1000 will be $1089 - 25 = 1064$.

Answer: 1064

3. **Solution:**



$$a^2 + b^2 = c^2 = 4r^2$$
$$a \cdot b = \tfrac{2}{9} p\, r^2$$
$$\tfrac{1}{2} a \cdot b = \frac{p\, r^2}{9}$$
$$\frac{a}{b} + \frac{b}{a} = \frac{18}{p}$$

Let $m\angle BAC = x$

$$\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{18}{p}$$
$$\frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{18}{p}$$
$$\sin x \cos x = \frac{p}{18}$$
$$\sin 2x = \frac{p}{9}$$
$$2x = 20.43^\circ$$

Answer: 10°

$$x = 10.2^\circ$$

4. **Solution:** $f(x) = x^3 + ax^2 + bx + c = 0$

$$a + b = -15$$
$$f(1) = 0 \Rightarrow a + b + c = -1 \Rightarrow c = 14$$

Since product of the roots $(1)(-7)(r_3) = -14$

$$r_3 = 2$$

Answer: 2

5. **Solution:** The sum of the roots must be -10.
Therefore the absolute value of the sum is 10.

Answer: 10

6. **Solution:** Johnny runs $\frac{11}{25}$ of the way by the time Frankie jumps, so only $\frac{3}{25}$ remain. Since $\frac{3}{25}$ is $\frac{12}{100}$, the answer must be 12 mph.

Answer: 12 mph

7. **Solution:** Johnny lives 7 blocks from Frankie and can get there ${}_7C_3 = {}_7C_4 = 35$ ways. Likewise Frankie lives 7 blocks from the school and they can get there in ${}_7C_5 = {}_7C_2 = 21$ ways. Thus altogether there are $(35)(21) = 735$ ways.

Answer: 735

8. **Solution:** Note that the roots are $-r_3, -r_1, r_1, -r_3$ and that $r_3 = 3r_1$.
So the roots are $\pm r$ and $\pm 3r$. The sum of the products of the roots taken two at a time is $-10r^2$ and must equal -50.
Thus $-10r^2 = -50$
or $r = \pm\sqrt{5}$ and $3r = \pm 3\sqrt{5}$. Thus the product of the roots is $k = 225$.

Answer: 225

9. **Solution:** Let $k+1+k+2+\ldots+k+m = 5^7$

$$mk + (1+2+m)$$
$$mk + \frac{m(m+1)}{2} = 5^7$$
$$m(2k+m+1) = 2\cdot 5^7$$

Largest m $\quad m = 2\cdot 5^3 \qquad m = 250$

$$2k+m+1 = 5^4$$
$$k = 187$$

Answer: 250

10. **Solution:** Johnny moves at an angular velocity of $\frac{360}{6} = 60\frac{\text{deg}}{\text{min}}$ and Frankie at $\frac{360}{5} = 72\frac{\text{deg}}{\text{min}}$.
They will be in line when $72t - 60t = 180$.

$$12t = 180$$
$$t = 15$$

Answer: 15

11. **Solution:** $n^4 - 360n^2 + 400 = n^4 + 40n^2 + 400 - 400n^2$

$$= (n^2 + 20)^2 - 400n$$

$$= (n^2 + 20 + 20n)(n^2 + 20 - 20n)$$

Set $n^2 - 20n + 20 = 1$

$$n^2 - 20n + 19 = 0$$

$$(n-1)(n-19) = 0$$

$n = 1$ or $n = 19$ Thus the prime numbers are 41 and 761.

Answer: 20

12. **Solution:** Let $n^2 + n + 109 = (n + j)^2 = n^2 + 2jn + j^2$

$$n + 109 = 2jn + j^2$$

$n = \frac{109-j^2}{2j-1}$ Replacing j with 1, 2, 3, …, 10 yields 4 integer values which are 108, 35, 20, and 3.

Answer: 166

13. **Solution:** $x^2 - y^2 = 10^6 + 1$

$$(x-y)(x+y) = (10^2 + 1)(10^4 - 10^2 + 1) = (101)(9901)$$

$$x - y = 101$$

$$x + y = 9901$$

$$2x = 10002$$

$$x = 5001, \quad y = 4900$$

$$(x-y)(x+y) = 1(10^6 + 1)$$

$$x - y = 1$$

$$x + y = 10^6 + 1$$

$$2x = 10^6 + 2$$

$$x = 500{,}001 \quad y = 500{,}000$$

Answer: 5001

14. **Solution:** In base ten a zero will occur every time there are factors of 5 and 2. Two zeros would result when factors of 25 and 4 are present. Three zeroes would result when 125 and 8 are present, etc. Likewise, in base six, there will be a zero for each $3 \text{ x } 2$, two zeroes for $3^2 \text{ x } 2^2$, etc. In 100 there are: 33 factors of 3, 11 factors of 9, 3 factors of 27, 1 factor of 81, 47 zeroes

Answer: 48

15. **Solution:** Some reasonable answers are:

$$2^{23},\ 2^{11}\cdot 3,\ 2^7 \cdot 3^2,\ 2^5 \cdot 3^3,\ 2^5 \cdot 3 \cdot 5,\ 2^3 \cdot 3^2 \cdot 5,\ 2^2 \cdot 3 \cdot 5 \cdot 7$$

The smallest of these is $2^3 \cdot 3^2 \cdot 5$.

Answer: $2^3 \cdot 3^2 \cdot 5 = 360$