Part I: Multiple Choice (Answers)

1) **Solution:** Let x be the number of students who failed.

$$60x + 84(60 - x) = 80 \cdot 60$$
$$24x = 2400$$
$$x = 10$$

Answer: (e)

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2) **Solution:** The sum of two sides must be greater than the third.

Answer: (c)

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3) **Solution:** The slopes of the two graphs must be negative reciprocals.

Answer: (a)

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4) **Solution:**

$$s = rq$$
$$108 = r\left(\frac{9}{2}\right)$$
$$r = 24\text{ ft.}$$

Answer: (d)

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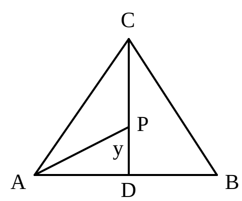
5) **Solution:**

$$y^2 + \left(\frac{x}{2}\right)^2 = (2y)^2$$
$$3y^2 = \left(\frac{x}{2}\right)^2$$
$$y = \frac{x}{2\sqrt{3}} = \frac{x\sqrt{3}}{6}$$



Answer: (a or c)

$$\overline{AP} = 2y = \frac{x\sqrt{3}}{3}$$

Note: The same answer was listed for a and c. Either correct answer was counted.

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6) **Solution:** Let $x = \log_3 16$ $\qquad$ $y = \log_2 27$

$$3^x = 16 \qquad 2^y = 27$$
$$x = \frac{\ln 16}{\ln 3} = \frac{4\ln 2}{\ln 3} \qquad y = \frac{\ln 27}{\ln 2} = \frac{3\ln 3}{\ln 2}$$
$$xy = \frac{4\ln 2 \cdot 3\ln 3}{\ln 3 \cdot \ln 2} = 12$$

Answer: (c)

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7) **Solution:**

$$9\tan\left(\frac{p}{4} - x\right) = \tan\left(\frac{p}{4} + x\right)$$

$$9\left[\frac{\tan\frac{p}{4} - \tan x}{1 + \tan\frac{p}{4}\tan x}\right] = \frac{\tan\frac{p}{4} + \tan x}{1 - \tan\frac{p}{4}\tan x}$$

$$9\left[\frac{1 - \tan x}{1 + \tan x}\right] = \frac{1 + \tan x}{1 - \tan x}$$

$$9(1 - \tan x)^2 = (1 + \tan x)^2$$

$$9(1 - 2\tan x + \tan^2 x) = 1 + 2\tan x + \tan^2 x$$

$$8\tan^2 x - 20\tan x + 8 = 0$$

$$(2\tan x - 1)(\tan x - 2) = 0$$

$$\tan x = 1/2; \quad \tan x = 2$$

Answer: (b) Product of the roots $= (\tan^{-1}(1/2))(\tan^{-1} 2) \approx .5133$

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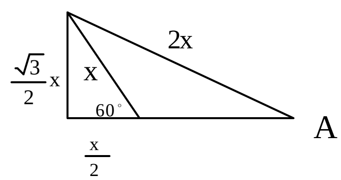
8) **Solution:**

$$\sin A = \frac{\frac{\sqrt{3}}{2}x}{2x} = \frac{\sqrt{3}}{4}$$

$$A = \sin^{-1}\frac{\sqrt{3}}{4} \approx 26^\circ$$



Answer: (d)

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9) **Solution:**

$$f(2x+1) = 4x^2 + 14x \Rightarrow f(x) = f\left(2\left(\frac{x-1}{2}\right) + 1\right)$$

$$f(x) = 4\left(\frac{x-1}{2}\right)^2 + 14\left(\frac{x-1}{2}\right)$$

$$= x^2 - 2x + 1 + 7x - 7$$

$f(x) = x^2 + 5x - 6$ which has roots of 1 and -6.

Thus the sum of the roots is -5.

Answer: (d)

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10) **Solution:**

$x^2 + (\cot A \cos A)x - 1 = 0$ with $0 < A < p/2$

$$x^2 + (\cos^2 A/\sin A)x - 1 = 0$$

$$(\sin A)\cdot x^2 + (\cos^2 A)\cdot x - \sin A = 0$$

$$x = \frac{-\cos^2 A \pm \sqrt{\cos^4 A + 4\sin^2 A}}{2\sin A}$$

$$x = \frac{-\cos^2 A \pm \sqrt{\cos^4 A + 4(1 - \cos^2 A)}}{2\sin A}$$

$$x = \frac{-\cos^2 A \ \pm \ \sqrt{\cos^4 A - \ 4\cos^2 A + 1}}{2\sin \ A}$$

$$x = \frac{-\cos^2 A \ \pm \ \sqrt{(\cos^2 A - \ 2)^2}}{2\sin A}$$

$$x = \frac{-\cos^2 A \ \pm \ (\cos^2 A - 2)}{2\sin A}$$

$$x_1 = \frac{-\cos^2 A \ + \ (\cos^2 A - 2)}{2\sin A} \ = \ \frac{-2}{2\sin A} = -\csc A$$

$$x_2 = \frac{-\cos^2 A \ - \ (\cos^2 A - 2)}{2\sin A} \ = \ \frac{-2\cos^2 A + 2}{2\sin \ A} = \ \sin A$$

Thus $x_1 + \ x_2 = \ \sin \ A - \csc A$

Answer: (b) (A quicker answer may be found by using the sum of the roots.)

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11) **Solution:**

$$e^2 x^{\ln x} \ = \ x^3$$
$$\ln\left(e^2 x^{\ln x}\right) = \ \ln\left(x^3\right)$$
$$\ln e^2 + \ln x^{\ln x} = 3\ln x$$
$$2 + \ln x \ln x = 3\ln x$$
$$(\ln x)^2 - 3\ln x + 2 = 0$$
$$(\ln x - 2)(\ln x - 1) = 0$$
$$\ln x = 2 \Rightarrow x = e^2$$
$$\ln x = 1 \Rightarrow x = e$$

Answer: (c) Thus the sum of the roots is $e^2 + e \approx 10.107$

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12) **Solution:** The roots will be real and unequal when $b^2 - 4ac > 0$. For the equation above, we have $k^2 - 4 \cdot 1 \cdot 1$, and so k must be greater than 2. The probability of this happening is $2/3$

Answer: (b)

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13) **Solution:**

$$\begin{array}{r} 2201_{\text{three}} \\ \times \ \underline{212}_{\text{three}} \\ 12102 \\ 22010 \\ \underline{1210200} \\ 2022012_{\text{three}} \end{array}$$

Answer: (d)

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14) **Solution:** $(a*b)*c = (b+1)*c = c+1$

$a*(b*c) = a*(c*1) = c+2$

Answer: (e) $c+1$ is never equal to $c+2$

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15) **Solution:** The total number of ways to choose 7 students is $7! = 5040$. The number of ways to choose a girl or boy alternating is $4\cdot 3\cdot 3\cdot 2\cdot 2\cdot 1\cdot 1 = 144$. Thus the quotient of the alternating ways divided by the total number of ways is $\frac{144}{5040} = \frac{1}{35}$.

Answer: (a)

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16) **Solution:** Let x be the required time in hours.

$$\frac{x}{2} - \frac{x}{5} = \frac{1}{2}$$

Answer: (b) $x = \frac{5}{3}$ or 1 hour and 40 minutes

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17) **Solution:** $\sqrt[3]{x+3p+1} = \sqrt[3]{x}+1$ Let $h = \sqrt[3]{x}$

$\sqrt[3]{x+3p+1} = h+1$

$x+3p+1 = h^3+3h^2+3h+1$

$h^3+3p+1 = h^3+3h^2+3h+1$

$3h^2+3h-3p = 0$

$h^2+h-p = 0$

$b^2-4ac = 1+4p \geq 0$ or $p \geq -\frac{1}{4}$

Answer: (b)

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18) **Solution:** $\tan\left(\tan^{-1}\frac{x}{10} + \tan^{-1}\frac{1}{x+1}\right) = \tan\frac{p}{4}$

$$\frac{\frac{x}{10}+\frac{1}{x+1}}{1-\frac{x}{10}\cdot\frac{1}{x+1}} = 1 \quad \text{or} \quad \frac{x}{10}+\frac{1}{x+1} = 1-\frac{x}{10}\cdot\frac{1}{x+1}$$

$x(x+1)+10 = 10(x+1)-x$

$x^2+x+10 = 10x+10-x$

$x^2-8x = 0$

Answer: (a) $x = 0, 8$

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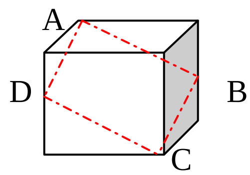
19) **Solution:** ABCD is a rhombus. $\overline{AC}$ has length $\sqrt{3}\,k$ and the diagonal $\overline{BD}$ has length $\sqrt{2}\,k$.



Area ABCD $= \dfrac{\sqrt{2}\,k \cdot \sqrt{3}k}{2} = \dfrac{\sqrt{6}\,k^2}{2}$

Answer: (b)

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20) **Solution:** Let x, $x+1$, and $x+2$ be three consecutive integers.

Case 1: If x is even, then $x+2$ is also even, so 4 divides the product.

Case 2: If x is odd, write x as $2k-1$. Two sub-cases exist.

2.1) If k is even, say $k = 2n$, then
$x+1=(2k-1)+1=2k=2(2n)$. So 4 divides the product.

2.2) If k is odd, say $k = 2n+1$, then
$x+1=(2k-1)+1=2k=4n+2=2(2n+1)$.
So 4 does not divide the product.

Thus the probability is $1/2(1+1/2)=3/4$.

Alternate Solution:

Any integer can be written in one of the six forms: $6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$, and $6k+5$. Let ? denote the product of 3 consecutive integers and consider the six cases below.

?			Probability M is divided by 12
$6k(6k+1)(6k+2)$	$12k(6k+1)(3k+1)$	Always divisible by 12	1
$(6k+1)(6k+2)(6k+3)$	$6(6k+1)(3k+1)(2k+1)$	Divisible by 12 when k is odd	$\frac{1}{2}$
$(6k+2)(6k+3)(6k+4)$	$12(3k+1)(2k+1)(3k+2)$	Always divisible by 12	1
$(6k+3)(6k+4)(6k+5)$	$6(2k+1)(3k+2)(6k+5)$	Divisible by 12 when k is even	$\frac{1}{2}$
$(6k+4)(6k+5)(6k+6)$	$12(3k+2)(6k+5)(k+1)$	Always divisible by 12	1
$(6k+5)(6k+6)(6k+7)$	$6(6k+5)(k+1)(6k+7)$	Divisible by 12 when k is odd	$\frac{1}{2}$

The probability of each case is $\frac{1}{6}$. Thus $\frac{1}{6}\left(1+\frac{1}{2}+1+\frac{1}{2}+1+\frac{1}{2}\right)=\frac{3}{4}$ is the probability that ? will be divisible by 12.

Answer: (e)

Part II: Integer Answer (15 Problems)

1) **Solution:** $2^{10} - 2^7 = 1024 - 128 = 896$

Answer: 896

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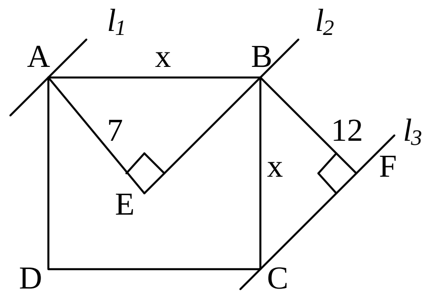
2) **Solution:** ?ABE $\cong$?BCF

$$\frac{7}{x} = \frac{\sqrt{x^2 - 12^2}}{x}$$
$$49 = x^2 - 12^2$$
$$x^2 = 193$$



Answer: 193

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3) **Solution:**

$$r_{avg} = \frac{2d}{t_1 + t_2} = \frac{2d}{\dfrac{d}{v+40} + \dfrac{d}{v-40}} = \frac{2}{\dfrac{2v}{v^2 - 40^2}} = \frac{v^2 - 40^2}{v}$$

$$393 = \frac{v^2 - 40^2}{v}$$
$$v^2 - 396v - 1600 = 0$$
$$(v - 400)(v + 4) = 0$$
$$v = 400$$

Answer: 400 mph

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4) **Solution:** $\frac{2}{9!} + \frac{2}{7!3!} + \frac{1}{5!5!} = \frac{20}{10!} + \frac{240}{10!} + \frac{252}{10!} = \frac{512}{10!} = \frac{2^9}{10!}$

$a = 9$ and $b = 10$ Thus $a + b = 19$

Answer: 19

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5) **Solution:** Note that $\frac{1}{\log_2 100!} = \frac{\ln 2}{\ln 100!}$, $\frac{1}{\log_3 100!} = \frac{\ln 3}{\ln 100!}$, etc

Thus $A = 100\left(\frac{\ln 2}{\ln 100!} + \frac{\ln 3}{\ln 100!} + \cdots + \frac{\ln 100}{\ln 100!}\right)$

$= 100\left(\frac{\ln(2 \cdot 3 \cdot 4 \cdot \cdots \cdot 100)}{\ln 100!}\right) = 100 \cdot 1 = 100$

Answer: 100

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6) **Solution:** Let $\frac{1}{n} = .abcdef\overline{abcdef}$

Then $10^6\left(\frac{1}{n}\right) = abcdef.\overline{abcdef}$

$999999\left(\frac{1}{n}\right) = abcdef$

$999999 = n\,(abcdef)$ Now consider the factors of 999999

. $999999 = 3\cdot 3\cdot 3\ \cdot 7\cdot 11\cdot 13\cdot 37$ The factors that satisfy the given conditions are 7, 13, 21, 39, 63, & 77 and their sum is 220.

Answer: 220

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7) **Solution:** Since $1+\sqrt{2}$ is a root, $1-\sqrt{2}$ must also be a root.

$(1+\sqrt{2})^4 + a(1+\sqrt{2}) + b = 0$

$(1-\sqrt{2})^4 + a(1-\sqrt{2}) + b = 0$

$(1+\sqrt{2})^4 - (1-\sqrt{2})^4 + 2\sqrt{2}\ a = 0$

$8\sqrt{2} + 16\sqrt{2} + 2\sqrt{2}\,a = 0$

$4 + 8 + a = 0$

$a = -12$, $b = -5$ Thus $b - a = 7$

Answer: 7

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8) **Solution:** Assume that side a is opposite $\angle A$.

$(a+b)(a-b) = c(b+c)$ or $a^2 - b^2 = c^2 + bc$

$a^2 = b^2 + c^2 + bc = b^2 + c^2 - 2bc\cos A$ **(**By Law of Cosines)

$bc = -2bc\cos A$

$\cos A = -1/2$ or $\angle A = 120^\circ$

Answer: 120°

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9) **Solution:** Let N^2 be a perfect square in the sequence. Then $N^2 - 1$ must be a multiple of 7. Thus $(N+1)(N-1) = 7m$. So either $(N+1)$ or $(N-1)$ is a multiple of 7.

Perfect Square	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	…	100^{th}
m	0	1	1	2	2	3		50
$N+1$		7		14		21		350
$N-1$	0		7		14			
N	1	6	8	13	15	20		349
N^2	1	36	64	169	225	400		349^2

Answer: $349^2 = 121801$

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10) **Solution:** $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}=\frac{a+b+c}{abc}=\frac{\text{sum of the roots}}{\text{product of the roots}}=\frac{4/k}{1/k}=4$.

Answer: 4

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11) **Solution:** $f(-1)=f(0)+1=k+1$

$f(-2)=k+1+4$

$f(-3)=k+1+4+7$

$\vdots$

$f(-50)=k+1+4+7+\cdots+148=4000$

$k + \frac{(1+148)}{2}\cdot 50=4000 \quad \text{or} \quad k+3725=4000$

$k=275$

Answer: 275

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12) **Solution:** $(1+x+x^2)^{12} = [1+(x+x^2)]^{12}$

$= 1^{12}+12\cdot 1^{11}(x+x^2)+66\cdot 1^{10}(x+x^2)^2+210\cdot 1^9(x+a^2)^3+\cdots$

(no terms involving x^3)

$= 66\cdot 2x^3+220x^3 = 352x^3$

Answer: 352

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13) **Solution:** Note that there is an intersection at $x=0$. There can be no intersection beyond $x=200$ which is approximately 64π. Within each period $(0,2\pi)$, $(2\pi,4\pi)$, …, $(62\pi,64\pi)$ there must be 2 intersections. These 64 intersections plus the one at zero yields 65 intersections.

Answer: 65

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14) **Solution:** Let the 5 points be ABCDE. Note that to paint the 5 points in the prescribed fashion, 2 colors must be used twice and one color just once. Also, note that if point A is a fixed color, then there are only two ways to paint the star. Consider one such set of colors, where red is the single color used. Then if A is painted red there are only two ways to paint the rest of the star. A similar situation occurs if the points B, C, D, or E are painted with the red. For this set of colors there are 10 ways to paint the star. Since there are three different sets of colors, there are a total of 30 ways to paint the star.

Answer: 30

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15) **Solution:** Note that $7^4=2401=2400+1=24(100)+1$

Now any number of the form $(100t+1)^{10}$ is a number of the form $1000t+1$, so 7^{40} is of that form.

$7^{1166}=7^{1160+6}=7^{1160}\cdot 7^6=(7^{40})^{29}\cdot 7^6$

Which has the same last 3 digits as 7^6 or 649.

Answer: 649
