2004 State Mathematics Finals: Algebra I
Solutions

1. D) $3x - c = 12 \Rightarrow 3x = 12 + c \Rightarrow x = \dfrac{12+c}{3} = 4 + \dfrac{c}{3}$.

2. B) The function is a parabola with vertex (2,4), passing through (0,2). Thus $f(x) = a(x-h)^2 + k \Rightarrow 2 = a(0-2)^2 + 4 \Rightarrow -2 = 4(a) \Rightarrow a = -\dfrac{1}{2}$, so the function for this graph is $f(x) = -\dfrac{1}{2}(x-2)^2 + 4 = 4 - \dfrac{1}{2}(x-2)^2$.

3. C) $\dfrac{B(A-B) - A(A+B)}{A(A-B) + B(A+B)} = \dfrac{AB - B^2 - A^2 - AB}{A^2 - AB + AB + B^2} = \dfrac{-(A^2+B^2)}{A^2+B^2} = -1$

4. D) Let *n* and *m* be two natural numbers. Then $n \cdot m = 30 + n + m \Rightarrow n \cdot m - n = 30 + m \Rightarrow n(m-1) = 30 + m \Rightarrow n = \dfrac{30+m}{m-1}$. Making *m* as small as possible, $m = 2 \Rightarrow n = \dfrac{30+2}{2-1} = 32$.

5. A) This sequence "telescopes" if you multiply it our and rearrange the terms. It will look like:

$$\left(\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{3}\right) + \left(\frac{1}{2}\cdot\frac{1}{3} - \frac{1}{2}\cdot\frac{1}{5}\right) + \left(\frac{1}{2}\cdot\frac{1}{5} - \frac{1}{2}\cdot\frac{1}{7}\right) + \ldots + \left(\frac{1}{2}\cdot\frac{1}{49} - \frac{1}{2}\cdot\frac{1}{51}\right) + \left(\frac{1}{2}\cdot\frac{1}{51} - \frac{1}{2}\cdot\frac{1}{53}\right) =$$

$$\frac{1}{2} + \left(-\frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3}\right) + \left(-\frac{1}{2}\cdot\frac{1}{5} + \frac{1}{2}\cdot\frac{1}{5}\right) + \left(-\frac{1}{2}\cdot\frac{1}{7} + \frac{1}{2}\cdot\frac{1}{7}\right) + \cdots + \left(-\frac{1}{2}\cdot\frac{1}{51} + \frac{1}{2}\cdot\frac{1}{51}\right) - \frac{1}{2}\cdot\frac{1}{53} =$$

$$\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{53} = \frac{1}{2}\left(1 - \frac{1}{53}\right) = \frac{1}{2}\left(\frac{52}{53}\right) = \frac{26}{53}$$

6. B) If the two quadratic expressions are to be the same for all *x*, the it follows that $x^2 - 49 + x^2 - 14x + 49 - 2px + 14p = kx^2 - 14kx + 49k$, thus $k = 2$ and $49k = 14p \Rightarrow p = 7$.

7. B) The paths of the cars can be shown as two right triangles, each with legs 3 and 7, making 2x the distance between the cars, $x^2 = 3^2 + 7^2 \Rightarrow x^2 = 58 \Rightarrow x = \sqrt{58}$, so $2x = 2\sqrt{58}$.

8. C) Note that to get the next term in the sequence, you need to add the next power of two to the previous answer. So $1 = 0 + 2^0, 3 = 1 + 2^1, 7 = 3 + 2^2, 15 = 7 + 2^3$, so the next term is $15 + 2^4 = 31$.

9. D) $36=6^2, 100=10^2, 225=15^2$, while $25=5^2, 49=7^2, 169=13^2$, so it appears as if Tina is selecting the squares of composite numbers while rejecting the squares of prime numbers. Since $64=8^2, 81=9^2, 196=14^2$, all are squares of composites, so she should keep them all.

10. B) If $x=0$, then $4x^2+4y^2-16y=9 \Rightarrow 4y^2-16y=9 \Rightarrow 4y^2-16y-9=0 \Rightarrow (2y-9)(2y+1)=0$, so $y=4.5 \text{ or } -0.5$. The distance between the points (0,4.5) and (0,-0.5) is 5.

11. D) On a single flip of the coin the $P(H)=0.6, so\, P(T)=1-0.6=0.4$. There are three ways to flip exactly two heads, HHT, HTH, and THH. Each has $P(2H)=(.6)(.6)(.4)$, so the desired probability is $3(0.6)(0.6)(0.4)=0.432$.

12. D) $\sqrt{x}=\sqrt[3]{2} \Rightarrow \left(\sqrt{x}\right)^6=\left(\sqrt[3]{2}\right)^6 \Rightarrow x^3=2^2 \Rightarrow x^3=4$.

13. C) $x-y=c \text{ and } y=x^2-5 \Rightarrow x-\left(x^2-5\right)=c \Rightarrow x^2-x+(c-5)=0$. When the discriminant is zero, there is exactly one solution, so $(-1)^2-4(1)(c-5)=0 \Rightarrow 1-4c+20=0 \Rightarrow 4c=21 \Rightarrow c=5.25$.

14. A) $\dfrac{a}{3-\sqrt{7}}=\dfrac{a}{3-\sqrt{7}}\left(\dfrac{3+\sqrt{7}}{3+\sqrt{7}}\right)=\dfrac{a\left(3+\sqrt{7}\right)}{9-7}=\dfrac{a\left(3+\sqrt{7}\right)}{2}$.

15. A) When two lines do not intersect, they are parallel and have the same slope. Since $3x-5y=2 \Rightarrow 5y=3x-2 \Rightarrow y=\left(3/5\right)x-2/5$, so the slope is $3/5=0.6$.

16. B) $x=y^2+y+1 \Rightarrow x=\left(y^2+y+1/4\right)+1-1/4 \Rightarrow x=\left(y+1/2\right)^2+3/4$. Thus the vertex of this parabola is $\left(3/4,-1/2\right)$.

17. C) Suppose d = distance between cites A and B. Since rate x time = distance, or $time=\dfrac{\text{distance}}{\text{rate}}$, then the time from city A to B is $\dfrac{d}{60}$ and the time from B to A is $\dfrac{d}{40}$. Then $\dfrac{d}{60}+\dfrac{d}{40}=\dfrac{2d}{ave} \Rightarrow \dfrac{40d+60d}{60\cdot 40}=\dfrac{2d}{ave} \Rightarrow ave=\dfrac{2\cdot 60\cdot 40}{40+60}=\dfrac{4800}{100}=48$.

18. C) $x=3y-1 \Rightarrow x+1=3y \Rightarrow y=\dfrac{x+1}{3}$.

19. E) $x-\frac{a^2}{x+2}=0 \Rightarrow x(x+2)-a^2=0 \Rightarrow x^2+2x-a^2=0$. Using the quadratic formula, $x=\frac{-2\pm\sqrt{2^2-4(1)(-a^2)}}{2}=\frac{-2\pm\sqrt{4+4a^2}}{2}=\frac{-2\pm 2\sqrt{1+a^2}}{2}=-1\pm\sqrt{1+a^2}$.

20. D) Let x = rate to wash one car. Then $2\cdot 6x=15 \Rightarrow 12x=15 \Rightarrow x={}^{15}\!/_{12}=1.25$. So 9 people can wash $9\cdot 4\cdot(1.25)=45$ cars in the same amount of time.

21. E) Let s be the length of the sides of the square. Then $2s^2=a^2 \Rightarrow s^2=\frac{a^2}{2} \Rightarrow s=\frac{a}{\sqrt{2}}$ and the perimeter is $4s=4\left(\frac{a}{\sqrt{2}}\right)=4\left(\frac{a}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right)=\frac{4a\sqrt{2}}{2}=2a\sqrt{2}$.

22. A) $1+2+4+\cdots+2^{17}=2^{18}-1=262143$ pennies = \$2621.43 and \$2621.43 – 1800 = \$821.43.

23. D) $\frac{1}{10}+\frac{1}{20}+\frac{1}{40}+\frac{1}{80}+\cdots=\frac{1}{5}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)$, but since $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=c$, it follows that $\frac{1}{10}+\frac{1}{20}+\frac{1}{40}+\frac{1}{80}+\cdots=\frac{1}{5}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)=\frac{1}{5}c=\frac{c}{5}$.

24. A) Let M = Mary's age in 1993. Since Mary is twice as old as Ben, his age in 1993 is ${}^{M}\!/_{2}$. In 2000 Mary is 7 years older that Ben, so $M+7=\left({}^{M}\!/_{2}+7\right)+7 \Rightarrow {}^{M}\!/_{2}=7 \Rightarrow M=14$. Thus Mary was 14 in 1993, Ben was 7, and in 2004 Ben was 7 + 11 = 18.

25. D) $\left(\sqrt{x}-\sqrt{3}\right)\left(\sqrt{x}+\sqrt{3}\right)-\left(\sqrt{x}-\sqrt{2}\right)^2=x-3-\left(x-2\sqrt{2x}+2\right)=2\sqrt{2x}-5$.

26. A) Let x be the robber's time. Then $x-{}^{1}\!/_{4}$ = the sheriff's time. Using rate x time = distance, $16x=20\left(x-{}^{1}\!/_{4}\right) \Rightarrow 16x=20x-5 \Rightarrow 4x=5 \Rightarrow x={}^{5}\!/_{4}=1.25\,hr$ and $x-{}^{1}\!/_{4}=1\,hr=60\min$.

27. C) $f(x)=g(x) \Rightarrow 11-2x=x^2-4 \Rightarrow 0=x^2+2x-15 \Rightarrow 0=(x+5)(x-3)$. Thus $x=-5$ *or* $x=3$. So the points of intersection are $(-5,21)$ *and* $(3,5)$ and $(-5,21)$ is farthest from the origin.

28. C) $x^2-8>-2x \Rightarrow x^2+2x-8>0 \Rightarrow (x-2)(x+4)>0$. The values which make the expression on the left equal to zero, namely $x=2$ *and* $x=-4$ divide the number line into 3 regions; namely $x<-4, -4<x<2, x>2$. By checking values in each of these intervals, we see that when $x<-4$ *or* $x>2$ the inequality holds.

29. D) $x^{-2}+(+b)x^{-1}+ab=0 \Rightarrow (x^{-1}+a)(x^{-1}+b)=0 \Rightarrow x^{-1}+b=0$ or $x^{-1}+a=0 \Rightarrow x=\frac{-1}{b}$ *or* $x=\frac{-1}{a}$.

30. E) It can be shown that a, c, and e are sometimes false by letting $f(x)=x^2+1$. It can be shown too that b is false by letting $f(x)=0$.

31. D) $\left(\sqrt{5}-\sqrt{3}\right)^2=5-2\sqrt{15}+3=8-2\sqrt{15}$ which is irrational. While this is all we really need, it is probably worthwhile to show that the others are rational. First $\sqrt{1.\overline{777}\cdots}=\sqrt{\frac{16}{9}}=\frac{4}{3}$. The second, $\left(9.\overline{123}\right)^{-2}$ is a repeating decimal, hence rational. To find the fraction for $9.\overline{123}$, let
$N=9.\overline{123} \Rightarrow 1000N=9123.\overline{123} \Rightarrow 1000N-N=9{,}123.\overline{123}-9.\overline{123} \Rightarrow 999N=9114$, so
$N=\frac{9114}{999}$. While 1.5129 is a terminating decimal, there is no guarantee its square root is rational, but $1.5129=\frac{15129}{10000}=\frac{123^2}{100^2} \Rightarrow (1.5129)^{-1/2}=\left(\frac{123^2}{100^2}\right)^{-1/2}=\left(\frac{100^2}{123^2}\right)^{1/2}=\frac{100}{123}$.
Finally, $\frac{\left(\sqrt{2}+1\right)^2}{3+\sqrt{8}}=\frac{2+2\sqrt{2}+1}{3+2\sqrt{2}}=\frac{3+2\sqrt{2}}{3+2\sqrt{2}}=1$.

32. B) Since (0,0), (42,0) and (3,2) are points on the curve, $f(x)=a(x-0)(x-42)$, and $f(3)=2=a(3-0)(3-42) \Rightarrow 2=-117a \Rightarrow a=-2/117$. The maximum height occurs at the parabola's vertex, which is midway between the zeros, or $x=21$, so
$f(21)=-2/117(21)(21-42)=\frac{2\cdot 21^2}{117}\approx 7.538$.

33. B) Since $64=8^2$, then $64^x=8^3 \Rightarrow \left(8^2\right)^x=8^3 \Rightarrow 8^{2x}=8^3 \Rightarrow 2x=3 \Rightarrow x=3/2$.

34. D) $322+341+401=2114$.

35. C) The duckweed covers $5\,cm^2$ on June 1 and doubles on June 5, 9, 13, 17, … 29. On June 29th the duckweed covers $5\cdot 2^7 = 640\,cm^2$. On June 30th, the weed will cover $5\cdot 2^{7.25} \approx 761.09\,cm^2$.

36. E) The lengths of the sides of the triangle can be represented by 4x, 3x, and 25. So $(4x)^2+(3x)^2=25^2 \Rightarrow 16x^2+9x^2=25^2 \Rightarrow 25x^2=25^2 \Rightarrow x=5$, so the lengths of the legs are 15 cm^2 and 20 cm^2 and the area is $\frac{1}{2}\cdot 15\cdot 20 = 150\,cm^2$.

37. B) $2\oplus(3\oplus 5)=2\oplus\left(\frac{3\cdot 5}{3+5}\right)=2\oplus\frac{15}{8}=\frac{2.15/8}{2+15/8}=\frac{30/8}{31/8}=\frac{30}{31}$

38. B) $3(x\oplus 1)=2x\oplus 5 \Rightarrow 3\left(\frac{x\cdot 1}{x+1}\right)=\frac{2x\cdot 5}{2x+5} \Rightarrow (3x)(2x+5)=(10x)(x+1)$, so $6x^2+15x=10x^2+10x \Rightarrow -4x^2+5x=0 \Rightarrow x(-4x+5)=0$, so $x=0,\ or\ x=5/4$. However, $x=5/4$ is the only positive solution.

39. A) $2x+3y=c \Rightarrow 3y=c-3x \Rightarrow y=\frac{c-2x}{3}$, so $3x+4\left(\frac{c-2x}{3}\right)=1 \Rightarrow 9x+4c-8x=3 \Rightarrow x=3-4c$.

40. A) $\frac{1}{4}\cdot 3\cdot 10\cdot 12+\frac{1}{4}\cdot 3\cdot 11\cdot 12+\cdots+\frac{1}{4}\cdot 3\cdot 16\cdot 12+\frac{1}{4}\cdot 3\cdot 17\cdot 12$, which, by factoring equals $\frac{1}{4}\cdot 3\cdot 12(10+11+12+\cdots+16+17)=\frac{1}{4}\cdot 3\cdot 12(10+17)\cdot\frac{8}{2}=3\cdot 12\cdot 27=972$.