# Algebra II
# State Mathematics Contest Finals, May 5, 2005

1. Given that the points, (1, 4) (6, 12) and ($c$,10), are colinear, what is the value of $c$?

   a) 3.5 b) 4 c) 4.25 d) 4.75 e) 5

2. If $f(x) = f(x-1) + x + 1$ and $f(5) = 7$, then $f(7)$ =?

   a) 9 b) 14 c) 22 d) 25 e) none of these

3. The average of Karen's five tests, each of which counted 15% of her final grade, was 83. The remaining part of the final grade depends on her score on the End-of-Course Exam. What grade does Karen need on this exam to raise her average to an 85?

   a) 87 b) 91 c) 93.2 d) 95 e) 99

4. What is the sum of all the real solutions to the equation $\sqrt{4x + \sqrt{17x^2 + 2}} = x + 2$?

   a) 0 b) $\sqrt{2}$ c) $\sqrt{7}$ d) $\sqrt{2} + \sqrt{7}$ e) none of these

5. Given $\frac{x+a}{x+b} = c$, solve for $x$.

   a) $x = \frac{cb-a}{1-c}$ b) $x = c - \frac{a}{b}$ c) $x = \frac{cb+a}{c-1}$ d) $x = \frac{c-1}{cb-a}$ e) $x = \frac{cb+a}{c+1}$

6. If $f^{-1}\left(\frac{1}{x+1}\right)=2x-3$ then

a) $f(x)=2x-4$. b) $f(x)=2x-5$. c) $f(x)=\frac{x-5}{2}$.

d) $f(x)=\frac{x+2}{2x+2}$. e) $f(x)=\frac{2}{x+5}$.

7. Find the ones digit of $7^{7^7}$.

a) 1 b) 3 c) 5 d) 7 e) 9

8. Find $p$ if $\sqrt{\frac{x}{y}\sqrt{\frac{y^3}{x^3}\sqrt{\frac{x^5}{y^5}}}}=\left(\frac{x}{y}\right)^p$.

a) $\frac{5}{8}$ b) $\frac{8}{5}$ c) $\frac{1}{8}$ d) $\frac{8}{3}$ e) $\frac{3}{8}$

9. Determine the solution set to $\frac{x^2+1}{x+2}<\frac{x+5}{2}$.

a) $\{x\,|\,-1<x<8\}$ b) $\{x\,|\,(x<-1)\cup(x>8)\}$ c. $\{x\,|\,(x<-2)\cup(x>8)\}$

d. $\{x\,|\,(x<-2)\cup(-1<x<8)\}$ e. $\{x\,|\,(-2<x<-1)\cup(x>8)\}$

10. Find the area of a triangle whose vertices are the intersection of the following three lines:

$3x+4y=75$ $4x-3y=0$ $y=0$

a) $\frac{675}{8}$ b) 150 c) $\frac{375}{2}$ d) $\frac{25\sqrt{3}}{2}$ e) $\frac{75\sqrt{5}}{2}$

11. Find the average value of all the solutions to $x^2+x+1=|4x-1|$.

a) $-\frac{1}{2}$ b) $\frac{-3+\sqrt{2}}{2}$ c) 0 d) $\frac{\sqrt{2}}{4}$ e) $\frac{1}{2}$

12. Let $A$ and $B$ be positive real numbers, and define $A \oplus B = \dfrac{AB}{A+B}$. Solve $X^2 \oplus (2X \oplus 1) = \dfrac{1}{4}$.

a) $-\frac{1}{2}$ b) $\frac{1}{2}$ c) $\frac{3}{2}$ d) $\frac{2}{3}$ e) None of these

13. Let $\oplus$ be defined as in problem #12. Find the general expression for $A \oplus A \oplus A \oplus \cdots \oplus A$ where $n$ *A's* are being combined using $\oplus$.

a) $nA$ b) $\dfrac{A^{n-1}}{n}$ c) $\dfrac{A}{n}$ d) $\dfrac{A^n}{1 + A + A^2 + \cdots + A^{n-1}}$ e) None of these

14. Determine the value of $p$ where $\sqrt{p + \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}} = 7$.

a) 49 b) $\dfrac{7 - \sqrt{7}}{2}$ c) 42 d) 4 e) None of these

15. Find the product of all real solutions of $x^4 - 5x^2 - 24 = 0$.

a) –24 b) –8 c) –3 d) $\sqrt{24}$ e) None of these

16. The average speed of a biker over the whole 55 mile trail was 15 mph. On the trail's 30 mile uphill section the biker was averaging 10 mph. What was the average speed on the downhill section? (The trail was either uphill or downhill, there were no level sections.)

a) 20 mph b) 21 mph c) 25 mph d) 36 mph e) None of these

17. Let $f(x) = x^2 - 12x + 5$, $g(x) = x + a$ and $f(g(x)) = x^2 + c$. Find the value of $c$.

a) –31 b) –7 c) $a^2 - 12a$ d) $5 - 12a$ e) 5

18. Each day Chris is chided for not cleaning up his room, so he picks up approximately 10% of the items on the floor in the morning. (He always rounds off to the nearest whole number if his calculations result in a fraction.) However during the course of each day, 10 new items somehow end up on the floor. If he has a clean floor on Sunday morning, how many items will be on the floor by the end of the week on Saturday night?

a) 14 b) 42 c) 52 d) 63 e) None of these

19. Suppose we are given the graph of $y = f(x)$. Which functional expression below describes a graph that is reflected across the *x*-axis then shifted 3 units to the right and 5 units up?

a) $3 - f(x-5)$ b) $f(x+3)-5$ c) $f(3-x)+5$ d) $f(x-3)+5$ e) $5 - f(x-3)$

20. Which of the following recursive equations generates the sequence: $\{1, 2, 5, 8, 2, 5, 8, 2, \cdots\}$?

a) $x_{t+1} = 15 - x_t - x_{t-1}$ b) $x_{t+1} = 3x_t - 1$ c) $x_{t+1} = 3x_{t-1} + 2$

d) $x_{t+1} = \frac{9}{2}x_t - \frac{1}{2}x_t^2 - 2$ e) $x_{t+1} = \frac{1}{2}(9 + 5x_t - 9x_{t-1})$

21. Find the remainder when $3^{98}$ is divided by 5.

a) 0 b) 1 c) 2 d) 3 e) 4

22. A parabola with a vertical axis of symmetry passes through the points (0,7), (4,15), and (12,7). Find the two *x*-intercepts.

a) {-2, 14} b) {-1, 13} c) $\{6 - \sqrt{43}, 6 + \sqrt{43}\}$ d) $\{6 - \sqrt{56}, 6 + \sqrt{56}\}$

e) The parabola does not cross the *x*-axis

23. Given a circle centered at the origin, and line tangent to this circle. Find the *y*-intercept of that line if the point of tangency is $(\sqrt{3}, 2)$.

a) $2 + \sqrt{3}$ b) $\frac{2 + 3\sqrt{3}}{2}$ c) $\frac{7}{2}$ d) $\frac{7 + 2\sqrt{3}}{3}$ e) none of these

24. Each spring a 12 meter × 12 meter rectangular garden has its length increased by 2 meters but its width decreased by 50 centimeters. What will be the maximum attainable area of the garden?

a) 144 $m^2$ b) 176 $m^2$ c) 189 $m^2$ d) 200 $m^2$ e) 225 $m^2$

25. Suppose you rolled two standard dice and the total number of pips on the top faces was 6. What is the probability that one of the dice had 2 pips on its top face?

a) $\frac{1}{6}$ b) $\frac{11}{36}$ c) $\frac{1}{5}$ d) $\frac{2}{5}$ e) $\frac{1}{3}$

26. Find the solution set of $\log(x+1) + 2\log(x-1) = 0$.

a) $\left\{\frac{1+\sqrt{5}}{2}\right\}$ b) $\left\{0, \frac{1\pm\sqrt{5}}{2}\right\}$ c) $\left\{\frac{\sqrt{5}-1}{2}\right\}$ d) $\left\{\frac{2}{3}\right\}$ e) $\left\{\frac{1\pm\sqrt{-3}}{2}\right\}$

27. Let $\frac{x-3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$ be an identity in $x$. What is the value of $A+B$?

a) 0 b) 1 c) $\frac{7}{3}$ d) 2 e) None of these

28. Solve the equation, $12 + 64^{x} = 8^{x+1}$.

a) $x = \frac{1}{3}$ only b) $x = \frac{1}{3}$ or $\frac{\log_2 6}{8}$ c) $x = \frac{1}{3}$ or $\frac{\log_2 6}{3}$

d) $x = \log_2 12$ only e) $\frac{\sqrt{3}}{2}$ or $\frac{1}{3}$

29. If $f(x)=\dfrac{x^2-x+1}{x+1}$ and $i=\sqrt{-1}$, then what does $f(1-i)$ equal?

a) $-i$ b) $\dfrac{1-2i}{5}$ c) $\dfrac{1-3i}{5}$ d) $\dfrac{3-2i}{5}$ e) $1+i$

30. A function of the form $f(x)=\dfrac{a}{x+b}$ has the following properties $f(1)=3$ and $f^{-1}(5)=-1$. What is the value of $f(0)$?

a) $\dfrac{15}{4}$ b) $-\dfrac{3}{5}$ c) 4 d) $\dfrac{15}{7}$ e) None of these

31. Given $p^2-2p+3=0$, Determine the value of $p^4-4p^3+8p-2$.

a) $-2$ b) 3 c) 19 d) 21 e) 23

32. Solve the following system of linear equations and determine the value of *x*.

$x+y=7$ $x-2z=8$ $y+3z=5$

a) 6 b) –17 c) 20 d) 7.5 e) –3

33. Assume you start with \$16. A fair coin is flipped, if it comes up heads \$8 are added to that amount, otherwise half the money is lost. Find the probability that you will have less than \$16 after the fourth flip.

a) $\dfrac{3}{8}$ b) $\dfrac{7}{16}$ c) $\dfrac{1}{2}$ d) $\dfrac{9}{16}$ e) $\dfrac{5}{8}$

34. Which of the following expressions is equivalent to $\log_8(32x)-\log_4(8x)+\log_2(x)$?

a) $\log_2(3x)$ b) $\frac{1}{6}\log_2(2x^5)$ c) $\frac{1}{3}\log_2(6x^2)$

d) $\log_2(0.5x^6)$ e) $\frac{1}{2}\log_2(4x)$

35. The line $y = 12 - 2x$ intersects the parabola $y = 5 + 6x - x^2$ at two points. Find the distance between these two points.

a) $6\sqrt{5}$ b) 10 c) $5\sqrt{2}$ d) $8\sqrt{3}$ e) 13

36. *N* is a natural number with properties that $10 < N < 100$, if *N* is divided by 7 its remainder is 3, and if *N* is divided by 13 its remainder is also 3. Find the product of the digits of *N*.

a) 0 b) 9 c) 36 d) 42 e) 64

37. A street sign 20 feet from a lamp post casts a shadow 5 feet long. If the lamp post were 1 foot taller the shadow would only be 4 feet 8 inches long. How high is the lamp on the post?

a) 15 ft. b) $17\frac{1}{2}$ ft. c) $25\frac{1}{3}$ ft. d) 35 ft. e) 70 ft.

38. Find the equation of a line that passes through point, $(2, \sqrt{3})$ and has a 30º angle of inclination.

a) $y = \sqrt{3} \cdot (x - 1)$ b) $y = \sqrt{2} \cdot (x - 2) + \sqrt{3}$ c) $y = 2 \cdot (x - 2) + \sqrt{3}$

d) $y = \dfrac{\sqrt{3} \cdot (x - 1)}{3}$. e) $y = \dfrac{\sqrt{3} \cdot (x + 1)}{3}$.

39. Find the difference between the largest and smallest root of $p(x) = x^4 - 6x^3 - 2x^2 + 6x + 1$.

a) 6 b) $4 + \sqrt{10}$ c) $4 - \sqrt{10}$ d) $1 + \sqrt{38}$ e) $\dfrac{7 + \sqrt{23}}{2}$.

40. A rectangular computer image in enclosed with a two pixel wide frame composed of 504 pixels. If the width of the image is increased by 10% then the frame needs 524 pixels. How many pixels does the image have?

a) 3,600 pixels b) 3,721 pixels c) 3,800 pixels d) 15,200 pixels e) None of these.