Mathematics Contest Spring 2005

Part I: Multiple Choice Questions (20 Problems)

1. George's company was losing money, as a result George received a 25% pay cut. By what percentage must his new pay rate be raised to bring it back to the original level?

a. 25% b. 50% c. 100% d. $33\frac{1}{3}\%$ e. 40%

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2. Let # be the binary operation on the set of positive real numbers that satisfies the following: $(xy^2) \# y = x(y \# 1)$ and $(x \# 1) \# x = 1$
If $1 \# 1 = 1$, then what is $x \# y$?

a. xy b. $\frac{y}{x}$ c. $\frac{x}{y}$ d. x^2y e. xy^2

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3. The value of $\sqrt{16+\sqrt{16+\sqrt{16+\ldots}}}$ is

a. $2\sqrt{2}$ b. 4 c. 4.52 d. 8 e. $\frac{1}{2}+\frac{\sqrt{65}}{2}$

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4. If $x + y + z = 0$, then $x^3 + y^3 + z^3$ equals

a. 0 b. $3xyz$ c. $-3x^2y$ d. $3xy^2$ e. none of the above

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5. If $log_4(log_4(log_4(log_4(x)))) = 0$, what is the value of x?

a. 256^3 b. 4^{16} c. 2^{512} d. 256^4 e. none of the above

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6. The base of a regular pyramid is a square with side length 10 meters. If the total surface area of the four triangular sides of the pyramid (not including the base) is 320 square meters, what is the height of the pyramid?

a. $2\sqrt{39}$ b. $\sqrt{231}$ c. 16 d. 32 e. none of the above

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7. A triangle has side $a = \sqrt{7}$, the opposite angle $\alpha = 60^{o}$, and the sum of the two other sides is $b + c = 5$ Find the ratio of the longest to the shortest side of the triangle.

a. 1 b. $\sqrt{2}$ c. $\frac{3}{2}$ d. $\frac{\sqrt{7}}{2}$ e. 2

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8. What is the value of $log_2\left(7^{-log_7 0.125}\right)$?

a. 3 b. -3 c. $\frac{1}{3}$ d. 0.125 e. 8

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9. A person starting with \$256 makes 8 bets and wins exactly four times. The wins and losses occur in random order. If each wager is for half the money she has at the time of the bet, then the final result is

a. a loss of \$81 b. a gain of \$81 c. a loss of \$175 d. neither a loss nor a gain

e. a gain or a loss depending on the order in which the wins and losses occur.

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10. A six sided die has faces labeled 1 through 6. It is weighted so that a three is three times as likely to be rolled as a one; a three and a six are equally likely; and a one, a two, a four, and a five are equally likely. What is the probability of rolling a three?

a. $\frac{1}{6}$ b. $\frac{1}{3}$ c. $\frac{2}{3}$ d. $\frac{3}{10}$ e. $\frac{2}{5}$

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11. What is the area of a triangle with sides 7, 8 and 9?

a. $12\sqrt{5}$ b. 31.5 c. 35 d. $18\sqrt{3}$

e. cannot be determined

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12. Thirty-six students took the ACT, with a mean score of 25.5. The boys had a mean score of 23.5, while the girls had a mean score of 28. How many girls were in the group?

a. 20 b. 18 c. 16 d. 14 e. cannot be determined.

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13. If m and n are natural numbers and $4m$-$5n$ =1, what is the greatest common divisor of m and n?

a. 4 b. 5 c. 20 d. 1 e. cannot be determined.

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14 Assume that a computation using method A takes $8n^2$ seconds, where n is a natural number and represents the size of the input. Assume that method B performs the same computation in $64\, n \log_2 n$ seconds. Which is the largest interval for n where A performs faster than B?

a. $n \geq 44$ b. $n \geq 32$ c. $2 \leq n \leq 43$ d. $2 \leq n \leq 64$ e. $1 \leq n \leq 32$

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15. Suppose that two circles C_1 and C_2 in the plane have no points in common. Then

a. there is exactly one line tangent to both C_1 and C_2.

b. there are exactly two lines tangent to both C_1 and C_2.

c. there are exactly three lines tangent to both C_1 and C_2.

d. there are no lines tangent to both C_1 and C_2 or there are exactly two lines tangent to both C_1 and C_2.

e. there are no lines tangent to both C_1 and C_2 or there are exactly four lines tangent to both C_1 and C_2.

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16. Convert the base four numeral 123.12 to a base five numeral.

a. 102.141414…… b. 102.414141……. c. 102.122222…

d. 102.121212… e. none of these.

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17. If $f(x)$ is an invertible function, and $g(x) = 2f(x) + 5$, then what is $g^{-1}(x)$?

a. $2f^{-1}(x) + 5$

b. $2f^{-1}(x) - 5$

c. $\dfrac{1}{2f^{-1}(x)+5}$

d. $\dfrac{1}{2}f^{-1}(x) + 5$

e. $f^{-1}\left(\dfrac{x-5}{2}\right)$

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18 A parallelogram has vertices (0,0), $(1,\sqrt{3})$, (4,0). A fourth vertex and the area are given by

a. $(5,\sqrt{3})$ and $4\sqrt{3}$

b. $(5,\sqrt{3})$ and $5\sqrt{3}$

c. $(4,\sqrt{3})$ and 10

d. (4,1) and $5\sqrt{3}$

e. (16,3) and 8

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19. If 7 distinct fair 6-sided dice are rolled at the same time, what is the probability that the sum will be 10?

a. $\dfrac{7}{279936}$

b. $\dfrac{7}{23328}$

c. $\dfrac{1}{139968}$

d. $\dfrac{1}{11664}$

e. none of these

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20. If the letters a, *A, b, B, c,* and *C* are arranged at random in a row, what is the probability that the lower case letters appear in increasing alphabetical order?

a. $\dfrac{1}{6}$

b. $\dfrac{1}{2}$

c. $\dfrac{1}{720}$

d. $\dfrac{1}{36}$

e. $\dfrac{1}{30}$

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Part II: Integer Answer Questions (15 problems)

1. Suppose that the coordinates of A and D are (1,5) and (1,10) respectively and that $ABCD$ forms a square with the x coordinate of B greater than 1. If F has coordinates (10,0) what is the area of the triangle BFC?

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2. How many subsets of $\{ n \mid 0 < n < 150 \text{ and } n \text{ is a multiple of } 4 \}$ are also subsets of $\{ n \mid 0 < n < 150 \text{ and } n \text{ is a multiple of } 6 \}$?

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3. Suppose a rectangle has area 3 and a diagonal of length $\sqrt{10}$. What is its perimeter?

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4. Several people started with $400 each, and played a game with the following unusual rules. Each player pays $10 to the house at the beginning of each round. During each round, one active player is declared the loser, and he distributes all of his money in equal amounts to the remaining players. The loser must then leave, but all of the other players go on to the next round. The game is over as soon as only one player remains. At the end of the game, the surviving player was surprised to discover that he had exactly $400, equaling his starting amount. How many players were there at the beginning?

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5. The driving distance from NCSSM in Durham to Disney World is 638 miles. The price of gasoline is $1.93 per gallon. How much would the gasoline cost - to the nearest dollar – for a round trip in a car that gets 24 miles per gallon

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6. How many four-digit positive integers divisible by 7 have the property that, when the first and last digits are interchanged, the result is a (not necessarily four digit) positive integer divisible by 7?

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7. Two numbers are called "approximately equal" if their difference is at most 1. How many different ways are there to write 2005 as a sum of one or more positive integers which are all "approximately equal" to each other? The order of terms does not matter: two ways which only differ in the order of terms are not considered different.

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8. Find the radius of a circle inscribed in a triangle with sides 12, 35, and 37.

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9. Find the sum of all values of k for which $2x^3 - 9x^2 + 12x - k = 0$ has a double root.

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10. How many real numbers t are there, so that the polynomial $x^{10} + tx + 1 = 0$ has a real solution r and also has $1/r$ as a solution?

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11. Let $a_1, a_2, a_3, \ldots$ be a sequence of integers satisfying $a_{n-1} + a_n = 3n$ for all $n \geq 2$. If $a_1 = 100$, find a_{1000}.

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12. Suppose n is a positive integer with the property that there are exactly eight different positive integers m such that $\frac{n}{m}$ is an integer. If one of these eight numbers is $m = 75$ what is the largest possible value of n?

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13. Find the smallest positive integer m such that m is not a square, but in the decimal expansion of $\sqrt{m}$ the decimal point is followed by at least four consecutive zeros. What is the integer part of $\sqrt{m}$ for this value of m?

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14. Let f be the function defined by $f(x,y,z) = (x + y + z)(xy + xz + yz)/(xyz)$ for all positive real numbers x, y, and z. What is the smallest possible value of f?

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15. In how many different ways can \$100.00 be made from 5-cent, 10-cent, and 25-cent coins if it is required that exactly 1000 coins be used?
