

ALGEBRA II
State Mathematics Finals Contest, May 3, 2007

1. A computer is programmed to read a list of numbers. For each p in the list, the computer carries out the following instructions.

If $p \leq p^2$, then print * .

If $p > p^2$, then print * * .

For which of the following values of p will * * be printed?

- a. $-\sqrt{2}$.
 - b. $-\frac{1}{3}$.
 - c. $\frac{1}{3}$.
 - d. $\sqrt{2}$.
 - e. 5.
2. The three digit number $2A3$ is added to the number 326 to give the three digit number $5B9$. If $5B9$ is divisible by 9, then $A + B =$
- a. 5.
 - b. 6.
 - c. 7.
 - d. 8.
 - e. 9.
3. Find the minimum value of $\sqrt{x^2 + y^2}$ if $5x + 12y = 60$.
- a. $\frac{60}{13}$.
 - b. $\frac{13}{12}$.
 - c. 1.
 - d. 0.
 - e. None of the above.

4. Anita attends a baseball game in Atlanta and estimates that there are 50,000 fans in attendance. Bob attends a baseball game in Boston and estimates there are 60,000 fans in attendance. A league official who knows the actual numbers attending the two games notes that:
- The actual attendance in Atlanta is within 10% of Anita's estimate.
 - Bob's estimate is within 10% of the actual attendance in Boston.

To the nearest 1000, the largest possible difference between the numbers attending the two games is

- 10,000.
 - 11,000.
 - 20,000.
 - 21,000.
 - 22,000.
5. A standard digital clock displays hours with numbers ranging from one to twelve as opposed to the European digital clock that displays hours with numbers ranging from one to 24. The difference between the least and greatest numbers that are squares of integers that can be displayed on a standard digital clock is
- 1035.
 - 1056.
 - 1104.
 - 1125.
 - 1196.
6. John was contracted to work A days. For each of these A days that John actually worked, he received B dollars. For each of these A days that John did not work, he had to pay a penalty of C dollars. After the A days of contracted work was over, John received a net amount of D dollars for his work. How many of the A days of contracted work did John not work?
- $\frac{AB - D}{B + C}$.
 - $\frac{AB + D}{B - C}$.
 - $\frac{AB + D}{B + C}$.
 - $\frac{AB - D}{B - C}$.
 - None of the above.
7. If $f(x) = (x + 5)^2 + 8$, then what is the sum of the values of x for which $f(x) = 12$?
- 10.
 - 7.
 - 10.
 - 20.
 - 297.

8. If $f(1) = 2$ and $f(n+1) = (f(n))^2$, what is the value of $f(4)$?

- a. 4.
- b. 16.
- c. 64.
- d. 256.
- e. 65,536.

9. Let b be a positive number such that the system

$$\begin{cases} ax + 3y = 1 \\ 5x + ay = b \end{cases}$$

has an infinite number of solutions. Then, correct to the nearest hundredth, b equals:

- a. 0.60.
- b. 1.29.
- c. 1.67.
- d. 3.87
- e. no unique answer.

10. A collection of nickels, dimes and quarters contains 56 coins altogether. The total value of the coins is nine dollars, and the value of the quarters alone is twice the value of nickels and dimes taken together. The number of nickels in the collection is:

- a. less than 5.
- b. 5, 6 or 7.
- c. 7, 8 or 9.
- d. 10, 11 or 12.
- e. more than 12.

11. If $2^x = 3$, then 3^x is approximately equal to

- a. 5.2.
- b. 5.7.
- c. 6.2.
- d. 6.8.
- e. 7.1.

12. If the sum of the squares of the lengths of all the sides of a rectangle is 100, then the length of a diagonal of the rectangle is

- a. $2\sqrt{5}$.
- b. $2\sqrt{13}$.
- c. $4\sqrt{3}$.
- d. $5\sqrt{2}$.
- e. 10.

13. Let $f(x)$ be defined as the least integer greater than $\frac{x}{5}$. Let $g(x)$ be defined as the greatest integer less than $\frac{x}{5}$. What is the value of $g(18) + f(102)$?

- a. 21.
- b. 22.
- c. 23.
- d. 24.
- e. 25.

14. If $3x$, $\frac{3}{x}$, and $\frac{15}{x}$ are integers, which of the following must also be an integer?

- I. $\frac{x}{3}$ II. x III. $6x$

- a. I only.
- b. II only.
- c. III only.
- d. I and III only.
- e. II and III only.

15. For all values of x , if x is even, x^* is defined as $0.5x$; if x is odd, x^* is defined as $\frac{x}{3}$. What is the value of $\frac{(6a)^*}{9^*}$, where a is an element of the rational numbers?

- a. $2a$.
- b. $3a$.
- c. a^* .
- d. $(2a)^*$.
- e. $(4a)^*$.

16. Suppose that you are asked to change a twenty dollar bill and that you have 20 one dollar bills, 4 five dollar bills and 2 ten dollar bills at your disposal. If x , y , and z denote, respectively, the number of one dollar, five dollar, and ten dollar bills exchanged for the twenty dollar bill, how many different ordered triples (x, y, z) are possible?

- a. 6.
- b. 7.
- c. 8.
- d. 9.
- e. 10.

17. A wooden cube with volume 64 is sliced in half horizontally. The two halves are then glued together to form a rectangular solid which is not a cube. What is the surface area of this new solid?

- a. 128.
- b. 112.
- c. 96.
- d. 56.
- e. 48.

18. A function f from the integers to the integers is defined as follows:

$$f(n) = \begin{cases} n + 3 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}.$$

Suppose k is odd and $f(f(f(k))) = 27$. What is the sum of the digits of k ?

- a. 3.
- b. 6.
- c. 9.
- d. 12.
- e. 15.

19. Let $E(n)$ denote the sum of the even digits of n . For example, $E(5681) = 6 + 8 = 14$. Find $E(1) + E(2) + E(3) + \dots + E(100)$.

- a. 200.
- b. 360.
- c. 400.
- d. 900.
- e. 2250.

20. If $\frac{P(x)}{x-a} = Q(x)$, where $P(x)$ and $Q(x)$ are polynomials, then

- a. $P(a) = 0$.
- b. $x - a$ is a factor of $P(x)$.
- c. a is an x -intercept of the graph of $y = P(x)$.
- d. All of the above.
- e. Insufficient information to answer.

21. In magic squares, the sum of the numbers in each row, each column, and each diagonal is constant. For the given magic square find the value of B .

19	A	14
10	B	C
D	E	11

- a. 12.
b. 15.
c. 17.
d. 18.
e. 21.
22. For $f(x) = x^2 - 1$ and $g(x) = |2x + 3|$, which has the greatest value?
- a. $f(g(3))$.
b. $g(f(2))$.
c. $g(f(-1))$.
d. $f(g(-5))$.
e. $g(10)$.
23. Two integers are said to be partners if both are divisible by the same set of prime numbers. The number of positive integers greater than 1 and less than 25 that have no partners in this set of integers is
- a. 9.
b. 10.
c. 11.
d. 12.
e. 13.
24. When an expression such as $\frac{2}{x^2 - 1}$ is expressed as the sum $\frac{1}{x - 1} + \frac{-1}{x + 1}$, the fractions in the sum are called partial fractions. When $\frac{19x - 8}{2x^2 - x - 21}$ is decomposed into partial fractions, what is the sum of the numerators when each fraction is reduced to lowest terms?
- a. 5.
b. 7.
c. 9.
d. 12.
e. 14.

25. An 8-ft-by-8-ft area has been tiled with 1-ft-by-1-ft tiles. Two of the tiles were defective. What is the probability that the two defective tiles share an edge?

- a. $\frac{1}{9}$.
- b. $\frac{1}{16}$.
- c. $\frac{1}{18}$.
- d. $\frac{1}{24}$.
- e. $\frac{1}{32}$.

26. Where $A = 1, B = 2, \dots, Z = 26$, we define $A \sim B = 2, A \sim B \sim C = 6, A \sim C \sim L = 36$, and $A \sim B \sim C \sim \dots \sim Z = 26!$. What set of three letters equals 1001 under the \sim operation?

- a. C, K, W.
- b. C, Q, S.
- c. G, I, M.
- d. G, K, M.
- e. I, K, M.

27. If $3^x = 5$, then $3^{2x+3} =$

- a. 37.
- b. 75.
- c. 270.
- d. 325.
- e. 675.

28. A tire on a car has an outside diameter of 25 inches. When the radius has been decreased a quarter of an inch, the number of revolutions per mile will be increased by what %?

- a. 1.8%.
- b. 2.0%.
- c. 2.2%.
- d. 2.4%.
- e. 2.6%.

29. In square $ABCD$, X lies on \overline{DC} such that $DX : XC = 5 : 2$ and Y lies on \overline{BC} such that $BY : YC = 3 : 4$. The ratio of the area of $\triangle AXC$ to the area of $\triangle ABY$ is

- a. 2:7.
- b. 2:3.
- c. 3:4.
- d. 4:9.
- e. 9:16.

30. How many integers x satisfy the equation $(x^2 - x - 1)^{x+2} = 1$?

- a. 2.
- b. 3.
- c. 4.
- d. 5.
- e. None of these.

31. Some motels have balconies for each room. Often these balconies have no exterior supports. When too many people go out on these balconies at events such as family reunions, the balconies collapse. Builders need to determine how much weight a balcony will hold without collapsing. An experiment using pennies for people and a centimeter ruler can be used to simulate the situation described. In the experiment, data is collected by extending the ruler (representing the balcony) beyond the edge of a flat surface. Then pennies (representing people of equal weight) are stacked on the extended edge of the ruler until the ruler falls. The following data was collected for such an experiment.

Length of extension (cm)	2	4	6	8	10	12	14
# of pennies supported by ruler	100	55	25	15	12	6	2

Which of the following equations represents the “curve” of best fit for the data?

- a. $y = 181.54(.739)^x$.
- b. $y = -7.23x + 88.57$.
- c. $y = 1.01x^2 - 23.38x + 137$.
- d. $y = -7.35x + 90.97$.
- e. $y = 126.82 - 50.29 \ln x$

32. If one solution to the equation $2x^2 + (a - 4)x - 2a = 0$ is $x = -3$, what is the value of a ?

- a. 0.
- b. 2.
- c. 4.
- d. 6.
- e. 12.

33. A circle is centered at the vertex of the right angle of an isosceles right triangle. The circle passes through the trisection points of the hypotenuse of the triangle. If the length of the radius of the circle is 10, find the area of the triangle.

- a. 45.
- b. 60.
- c. 75.
- d. 90.
- e. 105.

34. If $f(2x) = \frac{2}{2+x}$ for all $x > 0$, then $2f(x) =$

- a. $\frac{2}{1+x}$.
- b. $\frac{2}{2+x}$.
- c. $\frac{4}{1+x}$.
- d. $\frac{4}{2+x}$.
- e. $\frac{8}{4+x}$.

35. How many ordered pairs (m, n) of positive integers are solutions to $\frac{4}{m} + \frac{2}{n} = 1$?

- a. 1.
- b. 2.
- c. 3.
- d. 4.
- e. More than 4.

36. Consider the equation $x^2 + kx + 1 = 0$. A single fair die is rolled to determine the value of the middle coefficient, k . The value of k is the number of dots on the upper face of the die. The probability that the equation will have real, unequal roots is:

- a. $\frac{1}{3}$.
- b. $\frac{2}{3}$.
- c. $\frac{1}{2}$.
- d. $\frac{3}{4}$.
- e. None of these.

37. There exist positive integers A , B , and C , with no common factor greater than 1, such that $A \log_{200} 5 + B \log_{200} 2 = C$. What is $A + B + C$?

- a. 6.
- b. 7.
- c. 8.
- d. 9.
- e. 10.

38. An open box is to be created from a nine-inch by twelve-inch piece of posterboard by cutting congruent squares from each corner and folding up the sides. The goal is to cut squares of the size that will produce the box with maximum volume. Determine that volume to the nearest cubic inch.
- a. 82.
 - b. 98.
 - c. 118.
 - d. 150.
 - e. 164.
39. Each orange tree in a California grove produces 600 oranges per year if no more than 20 trees are planted per acre. For each additional tree planted per acre, the yield decreases by 15 oranges. How many trees per acre should be planted to obtain the greatest number of oranges?
- a. 20.
 - b. 22.
 - c. 25.
 - d. 30.
 - e. 32.
40. Suppose you toss two fair six-sided dice. Which probability is the greatest?
- a. $P(\text{one die is even and the other is odd})$
 - b. $P(\text{at least one die is prime and the sum is odd})$.
 - c. $P(\text{the sum is even and the product is a multiple of 5})$.
 - d. $P(\text{"doubles" and the product is odd})$
 - e. $P(\text{the sum is greater than 9 and one die is less than 4})$.