

Algebra II
State Mathematics Contest Finals, May 1, 2008

1. Find the y value of the intersection of $y = 2x + 7$ and $4x + 3y = 4$.

$$y = 2x + 7 \Rightarrow 4x = 2y - 14$$

Substituting: $(2y - 14) + 3y = 4 \Rightarrow 5y = 18 \Rightarrow y = 3.6$

2. For what values of b will $|2x - 1| = x + b$ have a unique solution?

$$|2x - 1| = x + b \Rightarrow 2x - 1 = \pm(x + b) \Rightarrow x = b + 1 \text{ or } 3x = 1 - b$$

For unique solution: $b + 1 = (1 - b)/3 \Rightarrow 4b = -2 \Rightarrow b = -1/2$

3. Let operator " \diamond " be defined on the set $\{p, r, s, t, u\}$, and assume " \diamond " is associative and commutative. If $p \diamond r = s$, $p \diamond p = t$ and $r \diamond t = u$. Find $p \diamond s$.

$$p \diamond s \Rightarrow p \diamond (p \diamond r) \Rightarrow (p \diamond p) \diamond r \Rightarrow t \diamond r \Rightarrow r \diamond t = u$$

4. What is the sum of all the positive odd numbers from 1 to 2008?

There are 1004 odd numbers with average value of 1004. Answer = $1004^2 = 1,008,016$

5. In a two way election between HC and BO. 55% of the voters were women and of those 60% voted for HC. Only 62% of the men voted for BO. What percentage of the vote did HC get?

HC got 38% of male vote thus $0.55 \cdot 60\% + 0.45 \cdot 38\% = 50.1\%$

6. Solve $9^x - 3^x = 12$.

$$9^x - 3^x = 12 \Rightarrow (3^x)^2 - 3^x - 12 = 0 \Rightarrow 3^x = 4 \text{ or } -3 \Rightarrow x = \log_3 4$$

7. Which of the following expressions if any is different from the other 3? ($i = \sqrt{-1}, c \in \text{Reals}$)

Simplifying each by rationalizing all denominators we get:

a) $\frac{i+c(1+i)}{2}$ b) $\frac{i+c(1+i)}{2}$ c) $\frac{i+c(1-i)}{2}$ d) $\frac{i+c(1+i)}{2}$ **c) is different.**

8. The location of a robot on the Cartesian plane at time t is given by (x, y) , where $x = 1.6t + 2$ and $y = t^2 - 2t$. What is the distance between the robot's location at $t = 0$ and $t = 5$?

Location at $t=0$ is $(2, 0)$ and at $t=5$ is $(10, 15)$, thus distance is $\sqrt{8^2 + 15^2} = 17$.

9. Jeremy has a bicycle repair shop and needs to know how much to charge for labor. If he charges too much he will lose customers, if he charges too little he won't make much money. At the present, he charges \$40 per hour and has 15 hours of work a week. He knows that for every \$5 increase in the hourly rate his workload drops by 3 hours, and for every \$5 decrease his workload goes up by 3 hours. How much should Jeremy charge per hour to maximize his profits?

Let x be the amount he charges per hour. Hours worked = $15 - \frac{3}{5}(x - 40)$. Thus

Profits = $x(15 - 0.6(x - 40)) = x(39 - 0.6x)$. Its maximum is at $x = \frac{1}{2} \cdot \frac{39}{0.6} = \mathbf{\$32.5}$

10. If $x - y = 7$ and $x^2 + 3xy - 4y^2 = 14$, find the value of $x + y$.

$$14 = x^2 + 3xy - 4y^2 = (x + 4y)(x - y) = (x + 4y)7 \quad \text{thus } x + 4y = 2.$$

$$x - y = 7 \Rightarrow x = 7 + y \Rightarrow (7 + y) + 4y = 2 \Rightarrow y = -1 \text{ and } x = 6. \quad \text{Thus } x + y = \mathbf{5}.$$

11. A class with 7 women and 5 men has 4 students chosen at random. If all have an equal chance of being picked, what is the chance that the group will have 2 men and 2 women?

$$\binom{4}{2} \cdot \binom{5}{2} \cdot \binom{4}{11} \cdot \binom{7}{10} \cdot \binom{6}{9} = \frac{14}{33}$$

12. For what value of m will the triangle formed by the lines $y = 4$, $x = 0$ and $y = mx - 2$ have an area of 42?

Vertices are: $(0, 4)$, $(0, -2)$ and $(6/m, 4)$ with a right angle at $(0, 4)$.

$$\text{Area} = \frac{1}{2} \cdot 6 \cdot 6/m = 42 \Rightarrow m = \mathbf{3/7}.$$

13. Let $n\downarrow$ denote the largest prime number less than n and $n\uparrow$ denote the smallest prime number greater than n . Find the value of $((10 + 20\uparrow)\downarrow + 30)\uparrow$

$$((10 + 20\uparrow)\downarrow + 30)\uparrow \Rightarrow ((10 + 23)\downarrow + 30)\uparrow \Rightarrow (31 + 30)\uparrow \Rightarrow \mathbf{67}$$

14. Given three standard die. They are rolled and someone tells you that the total of the top faces is 13. What is the probability that any one die has the number five on the top face?

Outcomes: $\{(6,6,1)\dots(1,6,6), (6,5,2)\dots(2,5,6), (6,4,3)\dots(3,4,6), (5,5,3)\dots(3,5,5), (5,4,4)\dots(4,4,5)\}$
 Thus there are 21 possible different outcomes 12 have a five. **Answer 4/7.**

15. $\lfloor x \rfloor$ is the largest integer less than or equal to x , and $\lceil x \rceil$ is the smallest integer greater than or equal to x , if x is NOT an integer which of the following is FALSE.

$$\lceil -x \rceil = -\lfloor x \rfloor \text{ is false if } x = 1.1. \lceil -1.1 \rceil = -1 \text{ but } -\lfloor 1.1 \rfloor = -2.$$

16. Let $g(x) = x^2 + b \cdot x + c$ and $g(2) = -6$ determine $g(5)$.

$$g(2) = 4 + 2b + c = -6 \Rightarrow b = -5 - \frac{1}{2}c \Rightarrow g(5) = 25 + (-5 - \frac{1}{2}c)5 + c = \mathbf{-1.5c}$$

17. The expression reduces to which of the following number?

$$\text{Let } x = \sqrt{5 + \sqrt{5 + \sqrt{5 + \sqrt{5 + \dots}}}} \text{ then } x = \sqrt{5 + x} \Rightarrow x^2 - x - 5 = 0 \Rightarrow x = \frac{1 + \sqrt{21}}{2}$$

18. Find the area of a triangle whose vertices are (1, 3), (5, 7), and (9, -1).

Calculate areas of trapezoids under line segments ((1,3), (5,7)), (5,7)(9,-1) and (1,3)(9,-1).

$$\text{Triangle} = \frac{1}{2}(5-1)(7+3) + \frac{1}{2}(9-5)(-1+7) - \frac{1}{2}(9-1)(-1+3) = \mathbf{24}$$

19. The graphs of $y = 5x - x^2$ and $y = m \cdot x + 9$ ($m < 0$) share only one point. Find m .

$$5x - x^2 = m \cdot x + 9 \Rightarrow x^2 + (m - 5) \cdot x + 9 = 0 \Rightarrow (m - 5)^2 = 36 \Rightarrow m = -1.$$

20. Stefan's weekly allowance depends on whether he does his chores or not. If he does them, it is increased by 50%, but if he doesn't do them it is decreased by 50%. If his allowance was \$16 initially, and he did no work for the four consecutive weeks, how many consecutive weeks must he work in order for his allowance to exceed \$16?

Stefan's allowance after four weeks of loafing is \$1. His allowance afterwards is 1.5^t .

Solving for t where $1.5^t > 16$ implies $t > \log(16)/\log(1.5) = 6.83\dots$ **Thus answer = 7.**

21. Let $2^a + 2^b + 2^c = 42$ where $a \neq b \neq c$ and $a, b,$ and $c \in \mathbb{N}$. Determine $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

$$42 = 32 + 8 + 2 = 2^5 + 2^3 + 2^1 \Rightarrow 1/5 + 1/3 + 1/1 = \mathbf{23/15}$$

22. Let $x + y = A$ and $x^2 + y^2 = B$. Express $x^3 + y^3$ in terms of A and B .

$$A^3 = x^3 + 3x^2y + 3xy^2 + y^3 \text{ and } A \cdot B = x^3 + x^2y + xy^2 + y^3 \text{ Thus } \frac{1}{2}(AB - A^2) = x^3 + y^3.$$

23. A certain rectangle has the property that if its width and length are each increased by 2 meters the area increases by 13 square meters. What is the perimeter of this rectangle?

$$\text{Let } w = \text{width and } k = \text{length, then } (w + 2)(k + 2) = wk + 13 \Rightarrow \mathbf{2w + 2k = 9}$$

24. Simplify: $\text{Log}_2(7a) + \text{Log}_4(7a^2) + \text{Log}_8(7a^3) + \text{Log}_{64}(7a^6)$

$$= \text{Log}_2 7 + \text{Log}_2 a + (\text{Log}_2 7 + 2 \cdot \text{Log}_2 a)/2 + (\text{Log}_2 7 + 3 \cdot \text{Log}_2 a)/3 + (\text{Log}_2 7 + 6 \cdot \text{Log}_2 a)/6$$

$$= 2 \cdot \log_2 7 + 4 \cdot \log_2 a = \mathbf{2 \cdot \log_2(7a^2)}$$

25. Sarah has 22 coins in her purse that total \$10.50 in value. Assuming these coins are dollar coins, quarters, and nickels, and that she has at least one of each kind, how many dollar coins does she have?

There are either 5 or 10 nickels since $25 \mid 1050$, and 10 implies at least 23 coins.
 Let $x = \# \text{dollars}$, $y = \# \text{quarters}$, thus $x + y + 5 = 22$ and $x + 0.25y + 0.25 = 10.50$.
 $\Rightarrow x + y = 17$ and $4x + y = 41 \Rightarrow x = 8$.

26. Today is Thursday May 1. What day of the week will May 1, 2018 be?

May 1, 2018 will be $8 \times 365 + 2 \times 366 = 3652$ days later. $3652 \bmod 7 = 5$, thus **Tuesday**.

27. The following repeated decimal fractions are added together $3.\overline{145} + 4.\overline{154}$ to get

$$3.145454545\dots + 4.154154154\dots = 7.29960869960\dots = 7.\overline{2996086}$$

28. A rubber ball is dropped from a height of 4 meters onto a concrete floor. It bounces to 75% of its original height and drops back down, bouncing many times in ever smaller bounces. If it did this forever how much total distance did the ball cover?

$$4 + 8 \cdot 0.75 + 8 \cdot 0.75^2 + 8 \cdot 0.75^3 + \dots = 4 + 8 \sum_{k=1}^{\infty} 0.75^k = 4 + 8 \frac{0.75}{0.25} = \mathbf{28m}$$

29. Solve $\sqrt{2-x} + x = 1$. If there are multiple solutions, give the largest one.

NOTE: $x \leq 2$: $2 - x = (1 - x)^2 \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1 - \sqrt{5}}{2}$

30. Find the equation of a line that is tangent to circle $x^2 + 4x + y^2 + 6y = 12$ at point (1, 1).

Circle rewritten as $(x + 2)^2 + (y + 3)^2 = 25$ thus its center is $(-2, -3)$.
 Slope of line from center to $(1, 1) = 4/3$, thus slope of tangent line is $-3/4$.
 Equation of tangent line is $\mathbf{3x + 4y = 1}$.

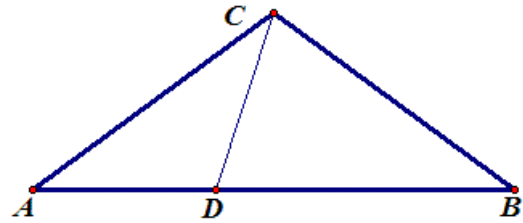
31. Given the function $f(x) = 0.5x^2 - 2x$, find the slope of the line connecting $(0, f(0))$ to $(6, f(6))$.

$$\text{Slope} = (f(6) - f(0)) / (6 - 0) = (6 - 0) / 6 = \mathbf{1}.$$

32. If $x^2 = x + 1$ then which of the following expressions is equal to x^6 ?

$$x^2 = x + 1 \Rightarrow x^3 = x^2 + x \Rightarrow x^3 = (x + 1) + x \Rightarrow x^3 = 2x + 1; \text{ similarly } x^4 = 2x^2 + x = 3x + 2, \\ x^5 = 3x^2 + 2x = 5x + 3, \text{ and } x^6 = 5x^2 + 3x = \mathbf{8x + 5},$$

33. An isosceles triangle ABC with point D on \overline{AB} , has $AC = BC = BD$ and $AD = CD$. If $AB = 2$, find the length of \overline{CD} .



Let $x = CD$

$$\text{Since } \triangle ABC \sim \triangle ABC \Rightarrow CD : AC = AC : AB \Rightarrow 2x = (AC)^2.$$

$$AD + BD = AB \Rightarrow x + AC = 2 \Rightarrow 2x = (2 - x)^2 \Rightarrow x^2 - 6x + 4 = 0 \Rightarrow x = 3 - \sqrt{5}$$

34. The perimeter of a right triangle is six times longer than its shortest side. What is the value of the ratio of the longer leg to the shorter one?

Let a , b , and c be the lengths of the sides of a right triangle in increasing order of size.
 $a + b + c = 6a \Rightarrow (b/a) + (c/a) = 5$ and $1 + (b/a)^2 = (c/a)^2 \Rightarrow 1 + (b/a)^2 = (5 - b/a)^2$
 Thus $1 = 25 - 10(b/a) \Rightarrow (b/a) = \mathbf{2.4}$

35. The equation $||x + 5| - 3| = c$, where c is a real number has exactly three distinct solutions. Find the value of c .

$$||x + 5| - 3| = c \Rightarrow |x + 5| = 3 \pm c \Rightarrow x = -5 \pm (3 \pm c) \Rightarrow x = -2 \pm c \text{ or } x = -8 \pm c.$$

To have three solutions $-2 - c = -8 + c \Rightarrow c = \mathbf{3}$.

36. If $f(1 - t^{-1}) = (5t + 1)/t$ then $f(t) = ?$

Let $x = 1 - t^{-1}$ then $t^{-1} = 1 - x$.

$$f(1 - t^{-1}) = 5 + t^{-1} \Rightarrow f(x) = 5 + (1 - x) = 6 - x. \Rightarrow f(t) = 6 - t.$$

37. Abigale is 10 years older than Bertha who is twice as old as Charlotte who is as old Dorothy and Eva combined. If Abigale is 5 times as old as Eva, and Bertha is 23 years older than Dorothy, how old is Charlotte?

Let a, b, c, d and e be the ages of Abigale, Bertha, Charlotte, Dorothy, and Eve respectively. Then $a = b + 10, b = 2c, c = d + e, a = 5e$ and $b = d + 23$, i.e. $d = b - 23$ and $e = 0.2(b + 10)$.

$$\text{Thus } c = b - 23 + 0.2b + 2 \Rightarrow c = 1.2b - 21 \Rightarrow c = 1.2(2c) - 21 \Rightarrow c = 15 \text{ years.}$$

38. $P = P_0(1 + \alpha)^t$, determines the amount of money, P , in an account that accrues α interest, compounded annually over a period of t years. P_0 is the amount of money you start with. Which expression calculates the time it takes for your money to triple?

$$3 P_0 = P_0 (1 + \alpha)^t \Rightarrow t = \log(3) / \log(1 + \alpha)$$

39. Which of the following expression is NOT a factor of $x^6 - 64$.

$$x^6 - 64 = (x^3 - 8)(x^3 + 8) = (x - 2)(x^2 + 2x + 4)(x + 2)(x^2 - 2x + 4) \text{ Thus } x^2 + 4 \text{ not a factor.}$$

40. The following is a 5x5 Sudoku puzzle. The digits 1-5 appear exactly once in each row, each column, and each region. Which of the digits should be in the shaded square?

Hint: start by finding locations for 3, then 2, then 1. Finally fill cells with 4's and 5's.

