ALGEBRA II
STATE MATHEMATICS CONTEST FINALS
APRIL 30, 2009

1. Determine the largest prime number dividing $27! + 28!$.

a. 7 b. 23 c. 29 d. 53 e. None of these

2. A bike manufacturer sells a particular model to a wholesaler. The wholesaler marks the bike up 10% and sells it to the retailer. The retailer marks the bike up 25%, but then puts it on sale for 10% off. What percentage is the sale price in terms of the wholesale price?

a. 123.75% b. 125% c. 125.25% d. 127.5% e. 122.5%

3. Find the x^4 coefficient of $(x - 3)^9$.

a. $\frac{3^5 9!}{4!5!}$ b. $-\frac{3^5}{4!}$ c. $-\frac{3^5}{5!}$ d. $\frac{9!}{5!}$ e. None of these

4. The determinant of the following matrix is 2. Assuming all entries in the matrix are integers, what is $|a| + |b|$?

$$\begin{bmatrix} 3 & a \\ b & 11 \end{bmatrix}$$

a. 16 b. 32 c. 12 d. 14 e. cannot be determined

5. One root of $2x^2 + 15x + c$ is four times the other. Determine c.

a. 36 b. 9 c. $\frac{3}{2}$ d. 18 e. $-\frac{3}{2}$

6. A rectangle of paper is said to be size A0 if it has an area of 1 square meter and the ratio of the long side to the short side is exactly the same as the ratio of the long to the short side of the rectangle that results from cutting the paper in half with a cut perpendicular to the longer side. When A0 paper is cut in half this way, it is called A1 paper. Similarly, A2 paper is obtained from A1 paper, but cutting it in half with a cut perpendicular to the long side of the rectangular page. Continuing in this way, one obtains A3 paper, A4 paper, etc. What is the perimeter of A4 paper?

a. $\dfrac{2\sqrt[4]{2}+\sqrt[4]{8}}{4}$m b. $\dfrac{1+\sqrt{5}}{3}$m c. $\dfrac{2\sqrt[4]{2}+\sqrt{5}}{4}$m d. $\dfrac{\sqrt{2}+1}{2\sqrt[4]{2}}$m e. $\dfrac{1+\sqrt[4]{2}+\sqrt{5}}{3\sqrt{2}}$m

7. Find the largest integer n for which $\dfrac{40!}{2^n}$ is an integer.

a. 38 b. 20 c. 32 d. 40 e. 24

8. Which of the following statements hold regarding conic sections and their foci?

I. The foci of an ellipse must *always* lie on the minor axis of the ellipse.
II. In a hyperbola, the difference of the distances between any point on the hyperbola and the two foci is a constant.
III. The sum of the distances from any point on an ellipse to its foci is the length of the major axis.

a. I only b. III only c. I and II only d. II and III only e. All are correct

9. A positive integer is said to be a palindromic number if it is equal to itself when its digits are reversed. Let $P_2, P_3, P_4, \ldots$ denote the number of palindromic numbers with two digits, three digits, four digits, and so on. What is true regarding the following sequence of ratios?

$$\frac{P_3}{P_2}, \frac{P_4}{P_3}, \frac{P_5}{P_4}, \frac{P_6}{P_5} \cdots$$

a. The sequence always increases.
b. The sequence always decreases.
c. The sequence is constant.
d. The sequence is composed of a finite number of values that neither increase nor always decrease.
e. The sequence is composed of an infinite number of values that neither increase nor always decrease.

10. At Heidi High everyone in the chess club is in the math club. Half of the people in the engineering club are in the math club. One third of the math club members are also in the chess club. Exactly half of the chess club also belong to the engineering club. Half as many students are in all three clubs as there are in just the engineering club. If there are 60 members in the engineering club, how many students are in at least one of these three clubs?

a.75 b. 120 c. 135 d. 90 e. 100

11. Find $2x + 3y$ given that $x + y = 6$ and $x^2 + 3xy + 2y^2 = 60$.

a.14 b. 22 c. 18 d. 20 e. None of these

12. Mary is very superstitious about playing poker, especially when she and Frances are the only ones playing. They always use a standard 52-card deck. Mary never looks at her last card until it's time to see who wins. Frances shows her hand of $9\clubsuit, 9\spadesuit, 9\diamondsuit, 2\heartsuit, 5\heartsuit$ while Mary glances at her first four cards: $A\diamondsuit, 2\diamondsuit, 3\diamondsuit, 4\diamondsuit$. In order to win, Mary's fifth card must be a 5 or any diamond. What is the probability that Mary wins?

a. $\frac{43!}{11!32!}$ b. $\frac{11}{43}$ c. $\frac{10}{43}$ d. $\frac{9}{43}$ e. $\frac{43!}{9!34!}$

13. In the Grantwood subdivision all homes have identical lawns. Three friends decide to start a lawn-mowing service on Saturdays to make some extra spending money. Working together it takes them 3 hours to mow 6 lawns. Next Saturday they have to mow 21 lawns, so they've invited four friends to join them. If every helper works at the same rate and they begin working at 9:00 AM, when will they finish all these lawns?

a. 12:30 PM b. 1:30 PM c. 1:15 PM d. 12:45 PM e. 1:00 PM

14. Define

$$s = \sum_{d=0}^{\infty} 32(3/4)^d.$$

Which of the following is true of s?

a. $0 \leq s \leq 32$ b. $32 < s \leq 64$ c. $64 < s \leq 128$ d. $128 < s \leq 256$ e. $256 < s < \infty$

15. The values of x satisfying

$$|2 - |2x - 4|| = 1$$

are distinct and can be ordered as $a < b < c < d$. Which of the following inequalities holds?

a. $0 < a < b < c < d$

b. $a < 0 < b < c < d$

c. $a < b < 0 < c < d$

d. $a < b < c < 0 < d$

e. $a < b < c < d < 0$

16. The number of pairs of non-negative integers satisfying the inequality

$$25 < a^2 + b^2 < 50$$

is

a. 19 b. 15 c. 30 d. 21 e. 12

17. If

$$a^2 + b^2 = 7ab,$$

with a and b both positive, then which of the following is equal to

$$\log(\frac{1}{3}(a+b))?$$

a. $\frac{7}{3}(\log a + \log b)$ b. $\frac{7}{2}(\log a + \log b)$ c. $\frac{1}{3}(\log a + \log b)$ d. $\frac{1}{2}(\log a + \log b)$ e. $\frac{1}{3}\log(a+b)$

18. Suppose that $a_1, a_2, \ldots, a_5$ are positive, single-digit integers and that

$$\sum_{n=1}^{5} a_n 10^n = 624380.$$

Find

$$\sum_{n=1}^{5} (-1)^n a_{6-n}.$$

a. 13 b. -16 c. 7 d. -13 e. 16

19. Find the real number t for which

$$4^t \cdot 4^{t+1} \cdot 4^{t+2} \cdot 4^{t+3} \cdot 4^{t+4} = 512.$$

a. -1 b. $-\frac{1}{2}$ c. $-\frac{2}{9}$ d. $\frac{5}{9}$ e. $-\frac{11}{10}$

20. Find the constant term in the real-valued polynomial of smallest order with roots $2-i$, i and 3 and with leading coefficient is 1.

a. -15 b. -6 c. 6 d. 3 e. None of these.

21. Mel's backyard is shaped like a right triangle with side lengths 4ft, 5ft and $\sqrt{41}$ft. She wants to build a rectangular garden in her backyard. What is the largest possible area of such a garden?

a. 10 sq.ft. b. 2 sq.ft. c. $\sqrt{41}$ sq.ft. d. 5 sq.ft. e. 4 sq.ft.

22. Determine

$$i \cdot i^2 \cdot i^3 \cdots i^{50},$$

where $i^2 = -1$.

a. 1 b. i c. -1 d. $-i$ e. Cannot be determined.

23. Suppose that c is a positive number such that

$$c = \sqrt{c + \sqrt{c + \sqrt{c + \cdots}}}.$$

What is true of c?

a. $c = \dfrac{1+\sqrt{5}}{2}$ b. $c = 2$ c. $c = e$ d. No such c exists. e. this holds for all $c \geq 1$.

24. A parabola of the form $y = ax^2 + bx + c$ with $a > 0$ intersects the graph of

$$f(x) = \frac{1}{x^2 - 4}.$$

Which set describes the number of possible intersections of these graphs?

a. $\{0, 1, 2\}$ b. $\{0, 1, 2, 3, 4\}$ c. $\{1, 2, 3, 4\}$ d. $\{0, 2, 3, 4\}$ e. $\{2, 3, 4\}$

25. Find the remainder when

$$x^{16} + 2x^{15} + 3x^{14} + \cdots 15x + 16$$

is divided by $2x - 2$.

a. 272 b. 68 c. -32 d. 136 e. -136

26. The function

$$f(x) = \frac{3x-2}{x+4}$$

has an inverse that can be written in the form $f^{-1}(x) = \dfrac{x+b}{cx+d}$. Find d.

a. $\frac{3}{4}$ b. -3 c. 4 d. -2 e. $-\frac{1}{2}$

27. Over the next three years BigCorp Holdings plans to cut its workforce by 13% from its current value. To make the transition easier, the boss wants to layoff p percent of the workforce this year; next year $p+1$ percent of the remaining workforce will be laid off and the year after that $p+2$ of the remaining workforce will be laid off. If x denotes the percent (as a decimal) that should be laid off in the first year, which of the following equations hold for x?

a. $x(x+1)(x+2) = 13$

b. $x(x-1)(x-2) = 87$

c. $x(x+0.01)(x+0.02) = .13$

d. $3x + 0.03 = .13$

e. $(1-x)(.99-x)(0.98-x) = .87$

28. The equation of the line through the vertex of $y = x^2 - 14x + 51$ and perpendicular to the line through the origin and $(4, 3)$ has the form $Ax + By = 34$. Find $A + B$.

a. 12 b. -6 c. 29 d. -19 e. 7

29. Jose earned \$150 during his first week at his summer job. He decided to spend it all at the electronics store. He bought some CDs for \$10 each, some discount DVDs at \$7 each and some blank writable DVDs for \$0.50 each. He bought exactly 100 items and spent all his money (assume there is no sales tax). How many discount DVDs did he buy?

a. 8 b. 9 c. 10 d. 11 e. 12

30. Suppose that $f(x)$ is a function of the form:

$$f(x) = \frac{ax^8 + bx^6 + cx^4 + dx^2 + 15x + 1}{x}$$

If $f(5) = 2$ what is $f(-5)$?

a. -2 b. 28 c. 17 d. 13 e. -13

31. How far is the furthest point on $x^2 + y^2 + 8x - 6y = 0$ away from the origin?

a. 10 units b. 14 units c. 5 units d. 7 units e. Cannot be determined.

32. For positive real numbers a and b define $a \diamond b$ as $a \diamond b = \sqrt{a+b}$. Which of the following is/are true?

I. The operation $\diamond$ always returns a positive number.
II. The operation is commutative.
III. The operation is associative.

a. I only b. II only c. I and III only d. I and II only e. II and III

33. How many values of c ensure that $cx^2 + 2cx + 3 = 0$ has only one (unique) solution?

a. 0 b. 1 c. 2 d. 3 e. infinitely many c.

34. If $x > 0$ and $\log_a b = 4$ find

$$\frac{\log_a x}{\log_{ab} x}.$$

a. 5 b. $\frac{1}{4}$ c. $\frac{1}{5}$ d. 4 e. Cannot be determined

35. Which of the following is not a candidate for a rational root of the polynomial $p(x)$ according to the **rational root theorem**?

$$p(x) = 6x^7 - 3x^2 + x + 5$$

a. 1 b. -1 c. $\frac{-5}{3}$ d. 5 e. -6

36. Let r be a real number. For what real numbers x is $f(x) = \ln\left(\frac{1}{r^2 - x^2}\right)$ defined?

a. $0 < x < r$

b. $-r < x < r$

c. $r < x < \infty$

d. $0 < x < \infty$

e. $-\infty < x < -r$ or $r < x < \infty$

37. The Lucas numbers are defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ when $n > 1$. Find L_{10}.

a. 55 b. 113 c. 66 d. 123 e. 76

38. Determine the value of the continued fraction:

$$1 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{\ddots}}}}$$

a. $\sqrt{5}$ b. $\frac{1+\sqrt{5}}{2}$ c. $\sqrt{3}$ d. $\frac{1+\sqrt{3}}{2}$ e. $\frac{1-\sqrt{5}}{2}$

39. For integers n and m, let $gcd(n, m)$ denote the *greatest common divisor* of n and m. Compute

$$\sum_{n=1}^{30} gcd(n, 30).$$

a. 232 b. 245 c. 465 d. 66 e. 135

40. A female bee has both a mother and a father, but a male bee comes from an unfertilized egg and so has no father. How many ancestors (not including himself) does a male bee have going back to his great-great-great "grand-bees?"

a. 11 b. 5 c. 30 d. 19 e. 8