The Thirty-second Annual State High School Mathematics Contest

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NC STATE MATHEMATICS CONTEST
APRIL 2010

PART I: 20 MULTIPLE CHOICE PROBLEMS

(1) The sum of the first five terms of an arithmetic sequence is 40, and the sum of the first ten terms of the sequence is 155. What is the sum of the first fifteen terms of the sequence?

a) 195 b) 230 c) 345 d) 390 e) 780

(2) If $\dfrac{a}{b+c+d} = \dfrac{4}{3}$ and $\dfrac{a}{b+c} = \dfrac{3}{5}$, then the value of $\dfrac{d}{a}$ is

a) $\dfrac{7}{6}$ b) $\dfrac{6}{7}$ c) $-\dfrac{12}{11}$ d) $-\dfrac{11}{12}$ e) $\dfrac{15}{11}$

(3) If $x + \dfrac{1}{x} = 4$, then the value of $x^3 + \dfrac{1}{x^3}$ is

a) 52 b) 60 c) 64 d) 68 e) 76

(4) If $3\sin\theta + 4\cos\theta = 5$, then $\tan\theta$ is

a) 1 b) -1 c) $\dfrac{3}{4}$ d) $\dfrac{4}{3}$ e) 0

(5) If $x < 0$, then $|x - \sqrt{(x-1)^2}|$ equals

a) 1 b) $1 - 2x$ c) $-2x - 1$ d) $1 + 2x$ e) $2x - 1$

(6) Which of the following equations have the same graph?

$$\text{I. } y = x - 2 \qquad \text{II. } y = \frac{x^2-4}{x+2} \qquad \text{III. } (x+2)y = x^2 - 4$$

a) I and II only b) I and III only c) II and III only d) I and II and III e) None

(7) If a, b, and c are the roots of the equation $x^3 - 3x + 7 = 0$, compute the numerical value of $(a+1)(b+1)(c+1)$.

a) -9 b) -4 c) 4 d) 5 e) 11

(8) The quadratic function $f(x) = ax^2 + bx + c$ is known to pass through the points $(-1, 6), (7, 6)$, and $(1, -6)$. Find the smallest value of the function.

a) -36 b) -26 c) -20 d) -10 e) -6

(9) If $f(x) = x + 2$ and $g(x) = \sqrt[3]{x}$, then $f^{-1} \circ g^{-1}(2)$ is

a) 8 b) -6 c) 2 d) 6 e) -2

(10) How many integers x satisfy the equation $(x^2 - 5x + 5)^{x^2-9x+20} = 1$?

a) 2 b) 3 c) 4 d) 5 e) none of a) through d) is correct

(11) If $A = 20^\circ$ and $B = 25^\circ$, then the value of $(1 + \tan A)(1 + \tan B)$ is

a) $\sqrt{3}$ b) 2 c) $1 + \sqrt{2}$ d) 4 e) none of a) through d) is correct

(12) The base three representation of x is 12112211122211112222. Find the first digit (on the left) of the base nine representation of x.

a) 1 b) 2 c) 3 d) 4 e) 5

(13) Let $f(n) = \left(\frac{1+i}{\sqrt{2}}\right)^n + \left(\frac{1-i}{\sqrt{2}}\right)^n$, where $i^2 = -1$. Determine the value of $f(2006) + f(2010)$.

a) 0 b) $\frac{2i}{\sqrt{2}}$ c) i d) $\frac{2}{\sqrt{2}}$ e) $-\frac{2i}{\sqrt{2}}$

(14) A box contains 2 pennies, 4 nickels, and 6 dimes. Six coins are drawn without replacement, with each coin having equal probability of being chosen. What is the probability that the value of the coins drawn is at least 50 cents?

a) $\frac{37}{924}$ b) $\frac{91}{924}$ c) $\frac{127}{924}$ d) $\frac{132}{924}$ e) none of a) through d) is correct

(15) The edges of a regular tetrahedron with vertices A, B, C, and D each have length one. Find the smallest possible distance between a pair of points P and Q, where P is on the edge AB and Q is on the edge CD.

a) $\frac{1}{2}$ b) $\frac{3}{4}$ c) $\frac{\sqrt{2}}{2}$ d) $\frac{\sqrt{3}}{2}$ e) $\frac{\sqrt{3}}{3}$

(16) A circle with an area a is contained in the interior of a larger circle with an area $a+b$. If the radius of the larger circle is 3, and if $a, b, a+b$ is an arithmetic sequence, then the radius of the smaller circle is

a) $\frac{\sqrt{3}}{2}$ b) 1 c) $\frac{2}{\sqrt{3}}$ d) $\frac{3}{2}$ e) $\sqrt{3}$

(17) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(xy)$ for all real numbers x and y, and $f(7) = 7$. Find the value of $f(49)$.

a) 1 b) 49 c) 7 d) 14 e) none of a) through d) is correct

(18) Equilateral triangle $\triangle ABC$ is inscribed in a circle. A second circle is tangent internally to the circumcircle at T and tangent to sides AB and AC at points P and Q. Determine the length of the segment PQ if side BC has length 12.

a) 6 b) $6\sqrt{3}$ c) 8 d) $8\sqrt{3}$ e) 9

(19) Two points are picked at random on the unit circle $x^2 + y^2 = 1$. What is the probability that the chord joining the two points has length at least 1?

a) $\frac{1}{4}$ b) $\frac{1}{2}$ c) $\frac{3}{4}$ d) $\frac{1}{3}$ e) $\frac{2}{3}$

(20) Let x, y, and z be positive real numbers such that $x+y+z=1$ and $xy+yz+xz=\frac{1}{3}$. The number of possible values of the expression $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$ is

a) 1 b) 2 c) 3 d) more than 3 but finitely many e) infinitely many

PART II: 10 INTEGER ANSWER PROBLEMS

(1) How many times does the prime factor 7 appear in the prime factorization of

$$1001 \cdot 1002 \cdot 1003 \cdots 2009 \cdot 2010?$$

(2) A road construction unit is made up of a certain number of workers and a certain amount of equipment. Three units have paved 20 mi of a road in 10 days. How many additional units are needed if the remaining 50 mi of the road must be paved in 15 days?

(3) Find the positive integer n for which $\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \cdots + \lfloor \log_2 n \rfloor = 153$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

(4) Find the smallest positive integer n for which none of the following fractions

$$\frac{7}{n+9}, \frac{8}{n+10}, \frac{9}{n+11}, \ldots, \frac{31}{n+33}$$

is reducible.

(5) In trapezoid $ABCD$, side AB is parallel to side DC, and diagonals AC and BD intersect at P. If the area of $\triangle APB$ is 4 and the area of $\triangle DPC$ is 9, determine the area of the trapezoid $ABCD$.

(6) If x is measured in radians, how many roots are there to the equation $\sin x = \dfrac{x}{100}$?

(7) In what base is 221 a factor of 1215?

(8) Determine the area of the polygon whose vertices are all the points on the circle $x^2 + y^2 = 100$ where both coordinates are integers.

(9) How many ordered pairs of integer numbers (x, y) satisfy the equation

$$\arctan \frac{1}{x} + \arctan \frac{1}{y} = \arctan \frac{1}{10}?$$

Note: $\arctan z$ is the same as $\tan^{-1} z$.

(10) Find the product of all distinct real solutions of the equation

$$(x^2 - 3)^3 - (4x + 6)^3 + 216 = 18(4x + 6)(3 - x^2).$$

If this equation has any repeated solutions, use them only once in the product.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Find the sum of the real solutions of the equation

$$\log_2 \left(-x^2 + 7x - 10\right) + 3\sqrt{\cos\left(\pi\sqrt{x^2 + 7}\right) - 1} = 1.$$