Solutions

PART I: 20 MULTIPLE CHOICE PROBLEMS

(1) Answer: c). The sum of the first n terms of an arithmetic sequence is $S_n = \frac{n[2a+(n-1)d]}{2}$ where a is the first term and d is the common difference. Thus, $40 = S_5 = 5a + 10d$ and $155 = S_{10} = 10a + 45d$. Solving this system, we get $a = 2$ and $d = 3$. Therefore, $S_{15} = 345$.

(2) Answer: d). From the given equations we get $\frac{b+c+d}{b+c} = \frac{9}{20}$. This implies $\frac{d}{b+c} = -\frac{11}{20}$. From the last equation and $\frac{a}{b+c} = \frac{3}{5}$ we get $\frac{d}{a} = -\frac{11}{12}$.

(3) Answer: a). $64 = 4^3 = \left(x + \frac{1}{x}\right)^3 = x^3+3x+\frac{3}{x}+\frac{1}{x^3} = \left(x^3 + \frac{1}{x^3}\right)+3\left(x + \frac{1}{x}\right) = \left(x^3 + \frac{1}{x^3}\right)+12$. Thus, $x^3 + \frac{1}{x^3} = 64 - 12 = 52$.

(4) Answer: c). From $3\sin\theta + 4\cos\theta = 5$ we get $\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta = 1$. If we square the last equation, we get $\frac{9}{25}\sin^2\theta + \frac{24}{25}\sin\theta\cos\theta + \frac{16}{25}\cos^2\theta = 1$. Using $\sin^2\theta + \cos^2\theta = 1$, we get $\frac{16}{25}\sin^2\theta - \frac{24}{25}\sin\theta\cos\theta + \frac{9}{25}\cos^2\theta = 0$, which is equivalent to $(\frac{4}{5}\sin\theta - \frac{3}{5}\cos\theta)^2 = 0$. From the last equation we obtain $\tan\theta = \frac{3}{4}$.

(5) Answer: b). Since $x < 0$, we have $\sqrt{(x-1)^2} = |(x-1)| = -(x-1) = 1-x$. Then $|x - \sqrt{(x-1)^2}| = |x-(1-x)| = |2x-1| = -(2x-1) = 1-2x$.

(6) Answer: e). I: The graph is the straight line $y = x - 2$ with domain of all real numbers.
II: The graph is the straight line $y = x - 2$ with "hole" at $(-2, -4)$ and domain of all real numbers except $x = -2$.
III: The graph is the union of the graphs of $y = x - 2$ and $x = -2$ and domain of all real numbers.

(7) Answer: a). If a, b, and c are the roots of $x^3-3x+7 = 0$, then $a+b+c = 0$, $ab+ac+bc = -3$, and $abc = -7$. Thus, $(a+1)(b+1)(c+1) = abc + ab + ac + bc + a + b + c + 1 = -9$.

(8) Answer: d). If the points $(-1, 6)$, $(7, 6)$, and $(1, -6)$ lie on the graph of $y = ax^2 + bx + c$, then $6 = a - b + c$, $6 = 49a + 7b + c$, $-6 = a + b + c$. Solving this system, we get $a = 1, b = -6, c = -1$. The graph of $y = x^2 - 6x - 1$ is a parabola that opens upward and the vertex is $(3, -10)$.

(9) Answer: d). Since $f^{-1}(x) = x - 2$ and $g^{-1}(x) = x^3$, we have $f^{-1} \circ g^{-1}(2) = f^{-1}(g^{-1}(2)) = f^{-1}(2^3) = 6$.

(10) Answer: d). $(x^2 - 5x + 5)^{x^2-9x+20} = 1$ if

- $x^2 - 9x + 20 = 0$, i.e. $x = 4$ or $x = 5$; or
- $x^2 - 5x + 5 = 1$, i.e. $x = 4$ or $x = 1$; or
- $x^2 - 5x + 5 = -1$ and $x^2 - 9x + 20$ is even. Then $x = 2$ and $x = 3$ satisfy both conditions.

Hence, the equation has five integer solutions.

(11) Answer: b). From $1 = \tan(A + B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$ we get $1 - \tan A\tan B = \tan A + \tan B$. Then $(1 + \tan A)(1 + \tan B) = 1 + \tan A + \tan B + \tan A\tan B = 1 + 1 - \tan A\tan B +$

$\tan A \tan B = 2$.

(12) Answer: e). $x = (1\cdot 3^{19} + 2\cdot 3^{18}) + (1\cdot 3^{17} + 1\cdot 3^{16}) + \ldots + (2\cdot 3 + 2) =$ $(1\cdot 3+2)(3^2)^9 + (1\cdot 3+1)(3^2)^8 + \ldots + (2\cdot 3+2)$. Hence, the first digit (on the left) of the base nine representation of x is 5.

(13) Answer: a). Since $\left(\frac{1+i}{\sqrt{2}}\right)^4 = -1$, $\left(\frac{1-i}{\sqrt{2}}\right)^4 = -1$, $2006 = 501\cdot 4 + 2$, and $2010 = 502\cdot 4 + 2$, we have $f(2006) + f(2010) = (-1)^{501}\left(\frac{1+i}{\sqrt{2}}\right)^2 + (-1)^{501}\left(\frac{1-i}{\sqrt{2}}\right)^2 + (-1)^{502}\left(\frac{1+i}{\sqrt{2}}\right)^2 + (-1)^{502}\left(\frac{1-i}{\sqrt{2}}\right)^2 = 0$.

(14) Answer (c). We can choose 6 coins from 12 on $\begin{pmatrix} 12 \\ 6 \end{pmatrix} = 924$ ways. We will have "at least 50 cents" if:

- Six dimes are drawn; the number of ways to choose six dimes is $\begin{pmatrix} 6 \\ 6 \end{pmatrix} = 1$.
- Five dimes and only one other coin are drawn; the number of ways to do this is $\begin{pmatrix} 6 \\ 5 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = 36$.
- Four dimes and two nickels are drawn; the number of ways to do this is $\begin{pmatrix} 6 \\ 4 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 90$.

The probability that the value of the coins drawn is at least 50 cents is $\frac{127}{924}$.

(15) Answer: c). The distance between P and Q is the smallest when P and Q are the midpoints of AB and CD, respectively. The segments PC and PD are altitudes of the equilateral triangles ABC and ABD, respectively, so $\overline{PC} = \overline{PD} = \frac{\sqrt{3}}{2}$. Since $\overline{QC} = \frac{1}{2}$, applying the Pythagorean theorem to $\triangle CPQ$, we get $\overline{PQ} = \frac{\sqrt{2}}{2}$.

(16) Answer: e). Since $a, b, a+b$ is an arithmetic sequence, we have $b - a = (b+a) - b$, i.e. $b = 2a$. Then $9\pi = a + b = 3a$, which implies $a = 3\pi$. Thus, the radius of the smaller circle is $\sqrt{3}$.

(17) Answer: c). If $y = 1$, we have $f(x+1) = f(x)$ for every real number x. Hence, $f(49) = 7$.

(18) Answer: c). Let O and D be the points at which PQ and BC intersect the diameter AT. The triangles APQ and PTQ are equilateral with one side in common. Hence, they are congruent, and O is the center of the larger circle. Since the triangles ABC and APQ are similar, we have $\frac{\overline{PQ}}{\overline{BC}} = \frac{\overline{AO}}{\overline{AD}} = \frac{2}{3}$. Thus, $\overline{PQ} = 8$.

(19) Answer: e). Denote the first point that is picked by A. Let B and C be the points on the circle which are exactly 1 unit away from A. Then $\overline{AB} = \overline{AC} = \overline{OA} = \overline{OB} = \overline{OC} = 1$ where O denotes the center of the circle. The triangles AOB and AOC are equilateral and the arc BC has angle $120°$. Hence, two-thirds of the points on the circle are at least 1 unit away from A. Therefore, the probability we are looking for is $\frac{2}{3}$.

(20) Answer: a) Since $1 = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+xz) = x^2+y^2+z^2+\frac{2}{3}$, we get $x^2+y^2+z^2 = \frac{1}{3}$. Then $x^2+y^2+z^2-xy-yz-xz = 0$, i.e. $\frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2] = 0$. Hence, $x = y = z$, which implies that $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 1$.

PART II: 10 INTEGER ANSWER PROBLEMS

(1) Answer: 169. $1001 \cdot 1002 \cdot 1003 \cdots\ 2009 \cdot 2010 = \frac{2010!}{1000!}$. The number of times that the prime factor 7 appear in the prime factorization of 2010! is

$$\lfloor \frac{2010}{7} \rfloor + \lfloor \frac{2010}{7^2} \rfloor + \lfloor \frac{2010}{7^3} \rfloor + \lfloor \frac{2010}{7^4} \rfloor + \cdots = 287 + 41 + 5 + 0 + \cdots = 333.$$

The number of times that the prime factor 7 appear in the prime factorization of 1000! is

$$\lfloor \frac{1000}{7} \rfloor + \lfloor \frac{1000}{7^2} \rfloor + \lfloor \frac{1000}{7^3} \rfloor + \lfloor \frac{1000}{7^4} \rfloor \cdots = 142 + 20 + 2 + 0 + \cdots = 164.$$

Therefore, the prime factor 7 appears $333 - 164 = 169$ times in the prime factorization of $1001 \cdot 1002 \cdot 1003 \cdots\ 2009 \cdot 2010$.

(2) Answer: 2. Let x be the number of additional units that are needed. Then $\frac{20}{3 \cdot 10} = \frac{50}{(3+x) \cdot 15}$. Thus, $x = 2$.

(3) Answer: 42. Notice the following pattern: $\lfloor \log_2 1 \rfloor = 0$; $\lfloor \log_2 2 \rfloor = 1$, $\lfloor \log_2 3 \rfloor = 1$; $\lfloor \log_2 4 \rfloor = 2$, $\lfloor \log_2 5 \rfloor = 2$, $\lfloor \log_2 6 \rfloor = 2$, $\lfloor \log_2 7 \rfloor = 2$; $\lfloor \log_2 8 \rfloor = 3$, $\lfloor \log_2 9 \rfloor = 3$, ..., $\lfloor \log_2 15 \rfloor = 3$; etc. If $n = 2^k - 1$ for some positive integer k, then

$$S_n = \sum_{i=1}^{2^k - 1} \lfloor \log_2 i \rfloor = 1 \cdot 0 + 2 \cdot 1 + 4 \cdot 2 + 8 \cdot 3 + \cdots + 2^{k-1}(k-1).$$

Hence, $S_1 = 0$, $S_3 = 2$, $S_7 = 10$, $S_{15} = 34$, $S_{31} = 98$, $S_{63} = 258$. Thus, $5m + 98 = 153$, which implies $m = 11$. Therefore, $n = 31 + 11 = 42$.

(4) Answer: 35. The fractions can be written as $\dfrac{k}{k + (n+2)}$, where $k = 7, 8, \ldots, 31$. These fractions are not reducible if k and $n + 2$ are relatively prime for every $k = 7, 8, \ldots, 31$. Since 37 is the smallest positive integer number that is relatively prime with $7, 8, \ldots, 31$, we get $n + 2 = 37$. Hence, $n = 35$.

(5) Answer: 25. Let PP' and PP'' be the altitudes of the triangles APB and CPD respectively, and let A be the area of the trapezoid $ABCD$. Then

$$A = \frac{1}{2}(\overline{AB} + \overline{CD})(\overline{PP'} + \overline{PP''}) = 4 + 9 + \frac{1}{2}(\overline{AB} \cdot \overline{PP''} + \overline{CD} \cdot \overline{PP'}).$$

Sine $\triangle APB$ and $\triangle CPD$ are similar, we have $\overline{AB} \cdot \overline{PP''} = \overline{CD} \cdot \overline{PP'}$. Hence $A = 13 + \overline{AB} \cdot \overline{PP''}$. Since $\dfrac{4}{9} = \dfrac{\overline{AB}^2}{\overline{CD}^2}$, we have $\overline{AB} = \dfrac{2\overline{CD}}{3}$. Then $\overline{AB} \cdot \overline{PP''} = \dfrac{2\overline{CD} \cdot \overline{PP''}}{3} = \dfrac{2 \cdot 18}{3} = 12$. Therefore, the area of the trapezoid $ABCD$ is 25 square units.

(6) Answer: 63. Since $-1 \le y \le 1$, we have that $-100 \le x \le 100$. From $x = 0$ to $x = 100$, the line $y = \frac{x}{100}$ intersects the graph of $y = \sin x$ exactly 32 times ($\frac{100}{2\pi}$ is approximately 15.92). By symmetry, there are 32 intersection points when x is non-positive. Since we count $(0, 0)$ twice, the total number of intersections is 63.

(7) Answer: 9. The numbers 1215 and 221 in base b are $b^3 + 2b^2 + b + 5$ and $2b^2 + 2b + 1$, respectively. Notice

$$b^3 + 2b^2 + b + 5 = (2b^2 + 2b + 1)(\frac{1}{2}b + \frac{1}{2}) + (-\frac{1}{2}b + \frac{9}{2}).$$

In order $221_{(b)}$ to be a factor of $1215_{(b)}$, the expression $-\frac{1}{2}b + \frac{9}{2}$ must be 0 and $\frac{1}{2}b + \frac{1}{2}$ must be an integer. Hence, $b = 9$.

(8) Answer: 296. First we will calculate the area of the polygon in the first quadrant. The vertices $A(10, 0)$, $B(8, 6)$, $C(6, 8)$ and $D(0, 10)$ are the only vertices with integer coordinates that lie on the circle $x^2 + y^2 = 100$. Let B' and C' be the orthogonal projections of B and C, respectively onto the x-axis. Then the area of the polygon $OABCD$ is a sum of the areas of the trapezoids $OC'CD$ and $C'B'BC$ and the triangle $B'AB$. The area of $OABCD$ is $\frac{1}{2} \cdot 6 \cdot (10 + 8) + \frac{1}{2} \cdot 2 \cdot (8 + 6) + \frac{1}{2} \cdot 2 \cdot 6 = 74$. Hence, the area of the whole polygon is $4 \cdot 74 = 296$ square units.

(9) Answer: 4. Using $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ and $\tan(\arctan\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$, we have

$$\tan\left(\arctan\frac{1}{x} + \arctan\frac{1}{y}\right) = \tan\left(\arctan\frac{1}{10}\right)$$
$$\frac{\frac{1}{x} + \frac{1}{y}}{1 - \frac{1}{xy}} = \frac{1}{10} \Leftrightarrow (x - 10)(y - 10) = 101.$$

The following four ordered pairs of integer numbers are solutions of this equation: $(11, 111)$, $(111, 11)$, $(9, -91)$, $(-91, 9)$.

(10) Answer: 9. Let $a = x^2 - 3, b = -(4x + 6), c = 6$. Then $a^3 + b^3 + c^3 = 3abc$ which is equivalent to $\frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (a - c)^2] = 0$. Now we have

$$\frac{1}{2}(x^2 - 4x - 3)[(x + 4x + 3)^2 + (4x + 12)^2 + (x^2 - 9)^2] = 0$$
$$\frac{1}{2}(x^2 - 4x - 3)(x + 3)^2(2x^2 - 4x + 25) = 0.$$

The real solutions of the equation are: $2 - \sqrt{7}, 2 + \sqrt{7}$, and -3 (with multiplicity 2). Their product is 9.

TIE BREAKER PROBLEM

Answer: 3. We will find the domain of this equation. Since $-x^2 + 7x - 10 > 0$, then x must satisfy $2 < x < 5$. Also, $\cos\left(\pi\sqrt{x^2 + 7}\right) - 1 \geq 0$, which implies $\cos\left(\pi\sqrt{x^2 + 7}\right) = 1$, i.e. $\sqrt{x^2 + 7} = 2k, k \in \mathbb{Z}$. Since $\sqrt{x^2 + 7} \geq 0$, we have $\sqrt{x^2 + 7} = 2k, k = 0, 1, 2, \ldots$. Hence, $2 < \sqrt{4k^2 - 7} < 5$, where $k = 0, 1, 2, \ldots$. For $k = 2$ we have $x = 3$, and for $k \geq 3$ we have $x > 5$ which is not possible. Therefore, the domain of this equation is the set $\{3\}$. It is easy to check that $x = 3$ is a solution of this equation.