2000 Algebra II State Finals
Solutions Manual

1. (D) If a line is described as $Ax + By + C = 0$ and there is a point (x_0, y_0), then the distance from the point to the line is

$$\left|\frac{Ax_0 + Bx_0 + C}{\sqrt{A^2 + B^2}}\right|$$

Here, this simplifies to $\left|\frac{-3}{\sqrt{2^2+4^2}}\right| \approx 0.67$

2. (E) The length of the radius is CP, and $CP = \sqrt{(-3-7)^2 + (1-5)^2} = 2\sqrt{29}$

3. (B) Recognizing difference of squares, $x^4 - y^4 = (x^2)^2 - (y^2)^2$

$$\frac{x^4 - y^4}{x^2 - y^2} = x^2 + y^2 = \frac{45}{5} = 9$$

4. (D) To obtain a term with a c^2, there must be 4 factors of $\sqrt{c}$ (and 22 factors of $\sqrt{d}$) in the term. With the coefficients of $\sqrt{c}$ and $\sqrt{d}$ being 1, the coefficient in the expansion then becomes

$$(1)^4(1)^2\left(\frac{26!}{4!22!}\right) = 14950$$

5. (A) This requires rearranging sides and squaring.

$$(\sqrt{3x-2} + \sqrt{2x-3})^2 = (1)^2$$

$$2\sqrt{6x^2 - 13x + 6} = 6 - 5x$$

$$4(6x^2 - 13x + 6) = 36 - 60x + 25x^2$$

$$x^2 - 8x + 12 = 0$$

$$(x-6)(x-2)$$

$$x = 2, 6$$

Neither solution works when checked in the original equation. Thus, there are 0 real solutions.

6. (B) let $x =$ the amount of antifreeze to be drained off. When replaced by antifreeze, the percentage of the total volume that is antifreeze is now

$$\frac{(9)(0.3) - x(0.3) + x(1.0)}{9} = 0.65$$
$$x = 4.5$$

7. (A) This is a $d = rt$ problem. Let $c =$ cruising speed and $w =$ wind speed

$$\tfrac{1200}{c+w} = 5 \qquad \tfrac{1200}{c-w} = 6$$

$$\tfrac{1200}{5} = c + w = 240 \qquad \tfrac{1200}{6} = c - w = 200$$

From here, it's a system of equations. $c = 220, w = 20$

8. (A)

$$\log 3^{x+2} = \log 2^{2x-1}$$
$$(x+2)\log 3 = (2x-1)\log 2$$
$$x = \frac{\log 2 + 2\log 3}{2\log 2 - \log 3} = \frac{\log 18}{\log \frac{4}{3}}$$

9. (B) Grouping/associating must be done in order to reduce the left-hand side to the sum of 2 squared binomials.

$$4(x^2 - 4x + 4) + 9(y^2 - 6x + 9) = -61 + 16 + 81$$
$$\frac{(x-2)^2}{3^2} + \frac{(y-3)^2}{2^2} = 1$$

The $x - 2$ term and the $y - 3$ term tell us that the center is $(2, 3)$. Vertices of the major axis must lie on the line $y = 3$, and they are $(2 \pm 3, 3)$.

10. (A) For any quadratic $Ax^2 + Bx + C = 0$, the roots are imaginary only if $B^2 - 4AC < 0$. Here,

$$(5\sqrt{2})^2 - 4(k)(-3) < 0$$
$$k < \frac{-25}{6}$$

11. (B) From $200 \rightarrow 300$, there are 26 multiples of 4 and 11 multiples of 10. However, this overcounts 6 numbers that are multiples of both 4 and 10. Thus, there are $26 + 11 - 6 = 31$ numbers from $200 \rightarrow 300$ that are divisible by 4 or 10.

12. (D) This operation (@) is not commutative because $x@y = x^2 - y^2$ and $y@x = y^2 - x^2$, but $x^2 - y^2 = y^2 - x^2$ is not true for all integers x, y. This operation is not associative because $(x@y)@z = x^2 - y^2 - z^2$ and $x@(y@z) = x^2 - y^2 + z^2$, but $x^2 - y^2 - z^2 = x^2 - y^2 + z^2$ is not true for all integers x, y. This operation merely replaces x by x^2 and y by y^2 and then performs a subtraction of the two terms. Since subtraction is neither a commutative nor an associative operation, it makes sense that the @ operation should be neither.

13. (A) Let the first person sit down. The 4 remaining people of the same gender must be next to choose their seats to ensure that they sit together. The second person has 4 seats to choose from out of 9 remaining seats, while the third person has 3 seats to choose from 8 remaining seats, etc. When the first 5 people are seated, the remaining 5 people, who are of the opposite gender, are forced to sit together. So the probability that the first 5 people sit together is

$$1 \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} \cdot \frac{1}{6} = \frac{1}{126}$$

14. (A) Let $x^2 = b$. Then

$$b^2 - 16b - 80 = 0$$

$$b = \frac{16 \pm \sqrt{16^2 - 4 \cdot 1 \cdot (-80)}}{2} = 20 \text{ (-4 is extraneous)}$$

$$x = \pm\sqrt{b} = \pm 2\sqrt{5}$$

15. (E) When a die is rolled 3 times, there are $6 \cdot 6 \cdot 6 = 216$ possible outcomes. If a 1 is rolled first, then there are 10 possible outcomes for the next two digits such that the third digit is greater than the second, which is greater than the first. If a 2 is rolled first, then there are 6 possible outcomes for the next two digits where the digits increase. If a 3 is rolled first , there are 3 ways for the digits to increase, and if a 4

is rolled first, there is 1 way for digits to increase. So there are a total of $10 + 6 + 3 + 1 = 21$ total outcomes. Thus the probability is

$$\frac{20}{216} = \frac{5}{54}$$

None of the given choices are correct. *Query: The total number of outcomes where the digits are increasing is the sum of triangular numbers. Why?*

16. (C) If $f(g(x)) = x$, then $f(x)$ and $g(x)$ are inverse funcitons. To find the inverse of $y = -2x + 1$, switch x and y and solve for y. The result is that $y = g(x) = -\frac{1}{2}x + \frac{1}{2}$. The sum of the slope and y-intercept is $-\frac{1}{2} + \frac{1}{2} = 0$

17. (B) Plugging the first point, we get $y = 5 = c$. Then we get from the next two points

$$4a + 2b + c = 11$$

$$-(\underline{4a - 2b + c = 15})$$

$$4b = -4$$

$$b = -1$$

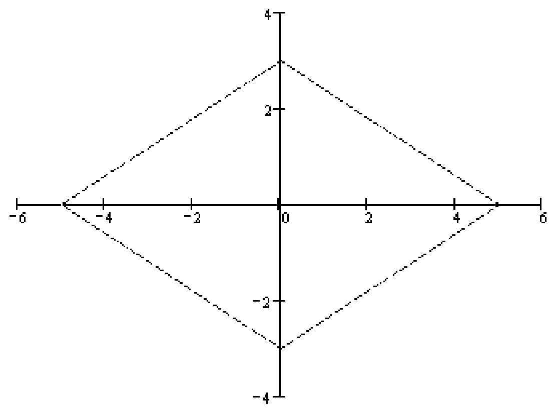
$$a = 2$$

$$a + b = 2 - 1 = 1$$

18. (E) The x in the denominator of $f(2x) = \frac{2}{2+x}$ represents half of the function's input. So $f(x) = \frac{2}{2+\frac{1}{2}x} = \frac{4}{4+x}$. $2f(x) = \frac{8}{4+x}$

19. (D) The number n will appear in the sequence from the $\frac{(n-1)n}{2} + 1$ term to the $\frac{n(n+1)}{2}$ term. Since n is an integer, the smallest n where $\frac{n(n+1)}{2} < 2000$ is $n = 62$. Thus, 62 will appear up until the 1953rd term. 63 appears from the 1954th term to the 2016th term, making the 2000th term 63.

20. (A) The area enclosed by the graph is a rhombus (shown in the figure below). The length of the diagonals of the rhombus are 10 and 6, so



the area is $\frac{6 \cdot 10}{2} = 30$

21. (B) Let $x = \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 \ldots}}}}$ Because the nested root repeats infintely

$$x^2 - 3 = x$$

$$x = \frac{1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-3)}}{2}$$

$$x = \frac{1 + \sqrt{13}}{2} \text{ because } x > 0$$

22. (C) Each term (except for the first two) is the sum of the previous two terms. The appropriate expression for the value of the nth term is $f(n) = f(n-1) + f(n-2)$

23. (D)

$$5^{-2} = 5^{3(3x-4)}$$

$$-2 = 9x - 12$$

$$x = \frac{10}{9}$$

24. (A)

$$\frac{\frac{-2}{x+1}}{\frac{5}{x} + 4}$$

$$= \frac{-2}{x+1} \cdot \frac{x}{5+4x}$$
$$= \frac{-2x}{4x^2+9x+5}$$

25. (C) If the transverse axis is horizontal, then the hyperbola must be of the form
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
The asymptotes will be of the form
$$y = \pm\frac{b}{a}x$$
The only hyperbola satisfying these conditions is
$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

26. (B) The equation of the circle can be found only by completing the square twice:
$$x^2 - 12x + y^2 + 6y = -29$$
$$(x^2 - 12x + 36) + (y^2 + 6x + 9) = -29 + 36 + 9$$
$$(x-6)^2 + (y+3)^2 = 4^2$$
It is now apparent that the center is at $(6, -3)$ and the radius is 4.

27. (C)
$$\cos(t) = \frac{2}{3}$$
For any value t, $\cos(t) = \cos(2\pi - t)$ Since $0 \leq t \leq 2\pi$, then $0 \leq 2\pi - t \leq 2\pi$. The fact that $-1 < \cos(t) < 1$ from the equation above only confirms the fact that a solution exists. There are only two solutions, and the sum of these two solutions is $t + (2\pi - t) = 2\pi$.

28. (D) Let $P(x)$ be the probability that at least one T-shirt is printed improperly. Then $1 - P(x)$ is the probability that every single T-shirt is printed properly. Since there is a 96% chance that a T-shirt is printed

properly, we can easily find the probability that every single T-shirt is printed properly. Given that the samplet set is 100 T-shirts,

$$1 - P(x) = (0.96)^{100}$$

Now solving for $P(x)$,

$$P(x) = 1 - (0.96)^{100}$$
$$\approx 0.983$$

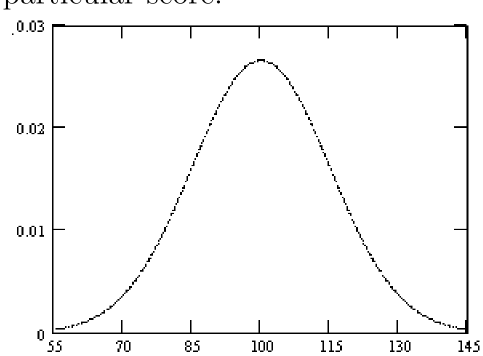
29. (E) In the first stretch of the trip, the driver has driven 120 miles and it has taken him $\frac{120}{50} = 2.4$ hours to do it. Thus, he has $250 - 120 = 130$ miles left to go and $4.5 - 2.4 = 2.1$ hours to with which to finish the trip. The average speed for the second stretch of the trip must then be $\frac{130}{2.1} = 61.9$ mph.

30. (B) The equation of the ellipse must be simplified by completing the square.

$$x^2 + 4y^2 + 2x - 24y + 33 = 0$$
$$(x+1)^2 + 4(y-3)^2 = 4$$
$$\frac{(x+1)^2}{2^2} + \frac{(y-3)^2}{1^2} = 1$$

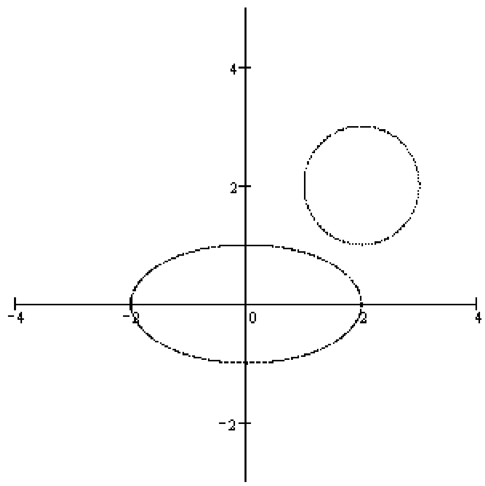
The minor axis is twice the semiminor axis, making the minor axis $2 \cdot 1 = 2$.

31. (C) From statistics, we know that about 68% of the data in a normal distribution of data will lie within one standard deviation of the mean. The figure below shows what percentage of the total tests received a particular score.



In this case, 68% of test scores will lie in the interval (85,115). It follows that 16% of the socres received a score below 85, and 16% of the scores received a score above 115. The percentage of scores that received a score greater than 85 is $68\% + 16\% = 84\%$.

32. (A) These two equations are the graph of a circle and an ellipse. The graph is as follows.



Since the graphs of both equations do not intersect, then are no solutions (x, y) that satisfy both equations.

33. (D) Multiply both sides by $(z+2)(2x-3)$. The resulting equation is now

$$A(2x-3) + B(z+2) = 5z - 11$$

When $x = \frac{3}{2}$, $B = \frac{\frac{15}{2}-11}{\frac{3}{2}+2} = -1$. When $x = -2$, $A = \frac{-10-11}{-7} = 3$.

$$A + B = 3 - 1 = 2$$

34. (D) Let $\log_b x = e$. Since b is a constant (not a variable), then in terms of x, the function returns y such that $b^y = x$. To find the inverse, switch x and y and solve for y. So

$$y = f^{-1}(x) = b^x$$

Making sense of this,

$$f(f^{-1}(x)) = \log_b b^x = x$$

35. (B) The total height in feet that the balloon rises is

$$80 + 80(0.9) + 80(0.9)^2 + \ldots$$

This is an infinite geometric series whose sum is

$$\frac{80}{1-(0.9)} = 800$$

36. (A) The lengths of the sides of a triangle are related to each to each other to their opposite angles through the Law of Sines. If a, b, and c are the sides of the triangle, with $a > b > c$, then $\frac{\sin 48^\circ}{c} = \frac{\sin 53^\circ}{b} = \frac{\sin 79^\circ}{a}$.

The first set of values, 48.0, 63.4, 51.6 satisfies the equation, with $c =$ 48.0 and

$$\sin 48^\circ = 48.0 \cdot \frac{\sin 53^\circ}{51.6} = 48.0 \cdot \frac{\sin 79^\circ}{63.4}$$

.

37. (D)

$$\begin{aligned} &\sin(x+30^\circ) + \cos(x+60^\circ) \\ &= \sin x \cos 30^\circ + \cos x \sin 30^\circ + \cos x \cos 60^\circ - \sin x \sin 60^\circ \\ &= \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x + \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x \\ &= \cos x \end{aligned}$$

38. (B)

$$\begin{aligned} &i^6 - i^{10} - i^{15} \\ &= i^2 - i^2 - i^3 \\ &= i \end{aligned}$$

39. (E)

$$\begin{aligned} &\frac{[(x+h)-(x+h)^2]-[x-x^2]}{h} \\ &= \frac{x+h-(x^2+2xh+h^2)-x+x^2}{h} \\ &\frac{h-2xh-h^2}{h} \\ &= 1-2x-h \end{aligned}$$

40. (E)

$$a = \{11, \frac{11}{1 \cdot 2}, \frac{11}{1 \cdot 2 \cdot 3}, \ldots\}$$
$$a_n = \frac{11}{n!}$$
$$a_6 = \frac{11}{6!}$$