SOLUTIONS

PART 1

1. Solution: total time $\frac{d}{40}+\frac{d}{60}$

total distance $= 2d$

$$\frac{2d}{\frac{d}{40}+\frac{d}{60}} = 48$$

Answer: 48 mph

2. Solution: axis $= \frac{-b}{2a}$

Distance from (u, v) to $\left(\frac{-b}{2a}, v\right)$ equals $\left|u+\frac{b}{2a}\right|$

Other point: $\frac{-b}{2a} \pm \left|u+\frac{b}{2a}\right| \to -\frac{b}{a}-u$ is on the graph

Answer: $\left(-\frac{b}{a}-u, v\right)$

3. Solution: $198 = 9 \cdot 11 \cdot 2$

Rule of 9: $x + y + 2 = 9$ or 18

Rule of 11: $x + 5 - y = 11$

$2a + 7 = 20, \quad \therefore x = 1, y = 6$

$N \div 198 = 77777$ Thus the last digit is 7

Answer: 7

4. Solution: $xy = (6-(x+y))^2 = 36-12(x+y)+(x+y)^2$

$xy = 36-12x-12y+x^2+2xy+y^2$

$x^2+xy+y^2 = 84$

$x^2-12x-12y+xy+y^2 = -36$

$12x+12y = 120$

$x+y = 10$

$x = 10-y$

Thus $100-20y+y^2+10y-y^2+y^2 = 84$

(2, 8) and (8, 2)

Answer: 16

5. Solution: Let $a = b = c = 1$

Answer: 256

6. Solution: Use right triangle with hypotenuse = 1.

Use right triangle with c = 1, a = x, and $b = \sqrt{1-x^2}$

Answer: $\dfrac{x}{\sqrt{1-x^2}}$

7. Solution: $1111111 = 239 \times 4649$

To get 239 use the calculator and set tables. Set the Table start at 1 and the change in table to 2. This will cause the table to go up by increments of 2. For "$y_1 =$" use 1111111/x. Calculate the table and scroll down the y_1 table until you find a number with no decimal.

Answer: $2+3+9=14$

8. Solution: $\log_4 \dfrac{1-x}{3-x} = \log_{.25} \dfrac{3+x}{2x+1} = y$

$$4^y = \frac{1-x}{3-x}$$

$$4^{-y} = \left(\frac{1}{4}\right)^y = \frac{3+x}{2x+1} \quad \text{or} \quad 4^y = \frac{2x+1}{3+x}$$

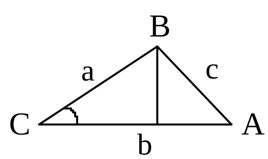
Thus $\dfrac{1-x}{3-x} = \dfrac{2x+1}{3+x}$

$$x^2 - 7x = 0$$

$x = 0, 7$ but x cannot be greater than 1

Answer: x = 0.

9. Solution:



$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area $a^2 + b^2 - c^2 = 2ab\cos C = \dfrac{ba\sin C}{2}$

$$\tan C = 4$$

$$\sec C = \sqrt{17}$$

Answer: $\sqrt{17}$

10. Solution: (1) $A = \frac{ab}{2} = a + b + c$

(2) $a^2 + b^2 = c^2$

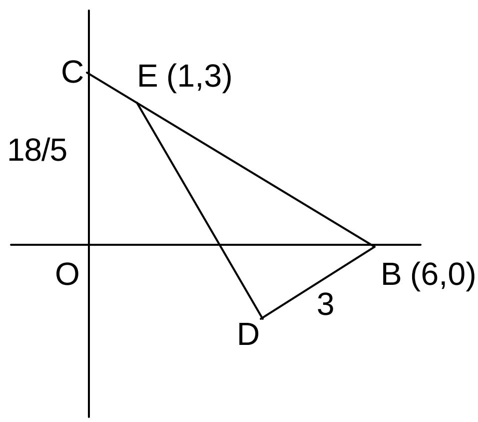
Solve (1) for c and substitute in (2) to obtain $(a-4)(b-4) = 8$

Thus $a = 6,\ b = 8$ or $a = 5, b = 12$

Hence there are 2. $\Rightarrow$ (6, 8, 10) and (5, 12, 13)

Answer: 2

11. Solution:



$\overline{DE} = 5$

$\overline{BE} = \sqrt{34}$

$\overline{OC} = \frac{18}{5}$

$\angle DBE = z$ and $\angle DBO = x$

$\tan z = \tan(x + y) = \frac{5}{3}$

$\tan y = \frac{\frac{18}{5}}{6} = \frac{3}{5}$

$\tan x = \frac{\tan z - \tan y}{1 + \tan z \tan y} = \frac{8}{15}$

Answer: 8 + 15 = 23

12. Solution: $5b + 1 \le 31a$

$43a \le 7b - 1$

$43(5b + 1) \le 31(7b - 1)$

$215b + 43 \le 217b - 31$

$74 \le 2b$

$37 \le b$

Answer: 37

13. Solution: Probability of 2 heads for 2-headed coin is 1 and for a fair coin is $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$. Since the probability of choosing one of the coins is 1 and the probability of the two headed coin is 4 times the other, the probabilities must be $\frac{1}{5}$ and $\frac{4}{5}$.

Answer: $\frac{1}{5}$

14. Solution: The next century will have $(5217)\cdot 7+5$ days. So 2101 begins on Saturday. Likewise 2201 on Thursday, 2301 on Tuesday, but the next century will be a leap century and so 2401 begins on Monday.

Answer: The pattern will repeat and so will never fall on Wednesday, Friday or Sunday.

15. Solution: Consider a parallelogram with sides 100 and 200 and diagonals c and d.

$$c^2 + d^2 = 2(100^2 + 200^2)$$
$$c^2 + (2\cdot 10\sqrt{61})^2$$
$$c^2 = 75600$$
$$c = 274.95$$

Answer: 274.95

16. Solution: Let R = radius of globe
r = radius of ball
b = distance from corner of box to center of globe

$$D = \sqrt{3}R = \sqrt{3}r + r + R$$
$$r = \left(2-\sqrt{3}\right)R = \left(2-\sqrt{3}\right)\cdot 10 = 2.7$$

Answer: 2.7

17. Solution: Since the left members are symmetric equations of x and y, they can be written as polynomials in the elementary symmetric equations:

$$c^2 + d - 2c = 8$$
$$2c^2 - 4d - 3c = 14$$

where $x+y=-c$, $xy=d$. The solutions of this system are (c, d) $(-2,0)$ and $\left(\frac{23}{6},\frac{35}{36}\right)$.

But x and y are the roots of the equation $u^2+cu+d=0$. From $u^2-2u=0$ we obtain the two solutions (x, y) = (0, 2) and (2, 0), and from the equation $36u^2+138u+35=0$ we obtain the solutions (x, y) = $(-0.273,-3.56)$ and $(-3.56,-0.273)$.

Answer: -1.833

18. Solution: Denote the tens thousands digits by a, the units digit by b, and the hundreds digit by (b + 1). Therefore, we have the equation $x^2 = 1000a + 100(b+1) + 10a + b$ or $(1010)a + (101)b = x^2 - 100$. This equation can also be written as $101(10a+b) = (x+10)(x-10)$. From the last equation, 101 is prime and $(10a+b)$ is at most in the nineties, owing to the nature of a and b. So, 101 must equal $(x+10)$. Therefore, x = 91, which then leads to $x^2 = 8281$.
Answer: 19

19. Solution: The second tallest player must be adjacent to the tallest player (either side). The third tallest person must be adjacent (either side) to the first two, and so on. Thus, the players can line up in $2^6 = 64$ different ways if left to right is considered different from the same order right to left.
Answer: 64

20. Solution: Note that $(\tan 1^\circ \cdot \tan 89^\circ)\cdot(\tan 3^\circ \cdot \tan 87^\circ)\cdot(\tan 5^\circ \cdot \tan 85^\circ)\cdot\ldots\cdot(\tan 91^\circ \cdot \tan 179^\circ) = 1$. This pairing will not include $\tan 45^\circ = 1$ or $\tan 135^\circ = -1$.
Answer: -1

PART 2

1. Solution: Rewrite the sum as:

$$1^2 + (3^2 - 2^2) + (5^2 - 4^2) + \cdots + (199^2 - 198^2) - 200^2$$
$$= 1 + (3-2)(3+2) + (5-4)(5+4) + \cdots + (199-198)(199+198) - 200^2$$
$$= 1 + 2 + 3 + \cdots + 199 - 200^2$$
$$= 19900 - 40000$$
$$= -20{,}100$$

Answer: $|-20{,}100|$

2. Solution: Note that F(0) = 2. Then find F(1) = 5 and F(2) = 26. F(5) $= 26^2 + 1 = 677$.
Answer: 677

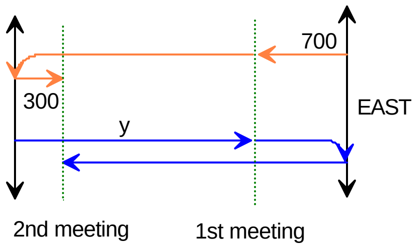
3. Solution: Let x = width of river
y = distance1st ship track
t = time sector

$x = y + 700$

$t \cdot (700) = y + 300$

$ty = 700 + (x - 300)$

Solving yields t = 2, y =1100, and x = 1800



Answer: 1800

4. Solution: $\cos 5x = \cos 5x + i \sin 5x = (\cos x + i \sin x)^5$

$\cos 5x = \cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^4 x + i($

$\cos 5x = \cos^5 x - 10 \cos^3 x(1 - \cos^2 x) + 5 \cos x(1 - \cos^2 x)^2$

$\cos 5x = 16 \cos^5 x - 20 \cos^3 x + 5 \cos x$

Answer: 1

5. Solution: $(1) + (2) - (4);\ v = -1$

$(2) + (3) - (5);\ x = 0$

By substitution $x = 0,\ y = 6,\ z = 7,\ u = 3$, and $v = -1$

Answer: 15

6. Solution: By law of cosines

$(120 - x)^2 = x^2 + 27^2 = 27 \cdot \cos 50^\circ$

Draw a triangle

Answer: 67 miles

7. Solution: $x^2 + y^2 - 10x = 25$

$(x - 5)^2 + y^2 = 25$

Under the reflection: Center $(5, 0) \rightarrow (-3, 8)$

Arbitrary Point $(8, 4) \rightarrow (1, 11)$

Locus is a circle with center $(-3, 8)$ and radius 5.

$(x + 3)^2 + (y - 8)^2 = 25$

$$x^2 + y^2 + 6x - 16y + 48 = 0$$
$$C + D + F = 6 - 16 + 48 = 38$$

Answer: 38

8. Solution:

$$2001 + n^2 = m^2$$
$$m^2 - n^2 = 2001$$
$$(m+n)(m-n) = 2001$$
$$(m+n)(m-n) = 3 \cdot 23 \cdot 29$$

$m+n$	2001	$29 \cdot 23$	$29 \cdot 3$	$23 \cdot 3$
$m-n$	1	3	23	29
n	1000	332	32	20

$\sum n = 1884$

Answer: 1384

9. Solution:

$$1.5x = 10x - a \cdot 10^n + a$$
$$a \cdot 10^n - a = 8.5x$$
$$a(10^n - 1) = \frac{17}{2}x$$
$$\frac{2}{17}(10^n - 1)a = x$$
$$\frac{2}{17}(999 \cdots 9)a = x$$

Since a is a single digit (and thus not divisible by 17), $999 \cdots 9$ must be divisible by 17:

$$2(58823529417647)a = x$$
$$1176470588235294a = x$$

Answer: 176470588235294 $\Rightarrow$ 17 digits

10. Solution: H ______ x . . . . . . W ______ y . . .1 . . . T

Note that from (2) and (4) that he walks 1 mile in 15 minutes so he walks 4 mi./hr. Note there are infinitely many solutions for the distance.

Answer: 4 miles per hour

11. Solution: Note that for rule #1 to work the base must be even. For rule #2 to work its base must be a multiple of 10. For rule #3 to work $12_b - b + 2 = b - 1 + 3$ must be divisible by 3^b which means $b - 1$ must be divisible by 3. The smallest base satisfying these conditions is 40.

Answer: 40

12. Solution: $x^2 + y^2 + z^2 = 12x + 12y + 12z$

$$(x-6)^2 + (y-6)^2 + (z-6)^2 = 108$$

x	16	16	16	12	12	12	8	8	4	0
y	8	8	4	12	12	0	8	4	4	0
z	8	4	4	12	0	0	−4	−4	−4	0

Answer: 10 solutions

13. Solution: Let a = b = 1

$$(1 * 1) + c = (1 * c) + 1$$

$c = 1 * c = c * 1$, thus 1 is the identity for $*$

Let c = 1

$$(a * b) + 1 = (b * 1) + a = b + a$$

$$a * b = a + b - 1$$

$$5 * (-3) = 5 + (-3) - 1 = 1$$

Answer: $|-3| = 3$

14. Solution: $P(x) = x^{100} - 4x^{98} + 5x + 6 = (x^3 - 2x^2 - x + 2) \cdot Q(x) + (ax^2 + bx + c)$

$$P(1) = 8 = a + b + c$$

$$P(-1) = -2 = a - b + c$$

$$P(2) = 16 = 4a + 2b + c$$

$$a = 1,\ b = 5,\ c = 2$$

Answer: 10

15. Solution: Note that the second hand must make one revolution and part of another. Also note that the angle of the second hand is the average of that of the minute hand and hour hand.

$$\therefore \frac{2p}{60} x = \frac{\frac{2p}{60^2} x + \frac{2p}{60^2 \cdot 12} x}{2} + 2p$$

$$x = \frac{86400}{1427} = 60.5466 \text{ seconds}$$

86400 + 1427 = 87827

Answer: 87827