1999 Geometry State Finals
Solutions Manual

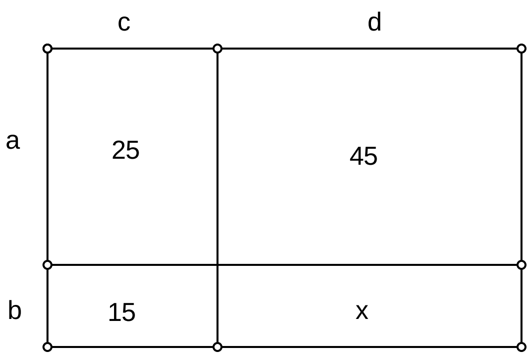
1. If $n =$ the number of sides, then

$$\frac{180(n-2)}{n} = 8\left(\frac{360}{n}\right)$$

$$180n - 360 = 2880$$

$$n = \mathbf{18}$$

2.



$$ad = 45$$

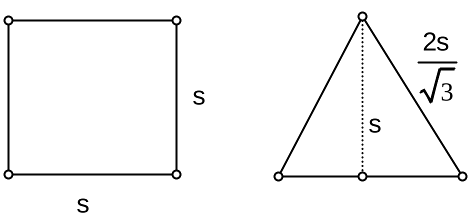
$$bc = 15$$

$$abcd = 675$$

$$ac = 25$$

$$bd = \frac{abcd}{ac} = \frac{675}{25} = \mathbf{27}$$

3.



$$k = \frac{\text{A of square}}{\text{A of triangle}} = \frac{s^2}{\frac{\left(\frac{2s}{\sqrt{3}}\right)^2\sqrt{3}}{4}} = \frac{s^2}{\frac{s^2\sqrt{3}}{3}} = \boldsymbol{\sqrt{3}}$$

4. The path of the tip of the hand is the same as the circumference. In 25 minutes, it will travel $\frac{5}{12}$ of the circumference, which is $\frac{5}{12}\cdot 2p(12) = \mathbf{10}\boldsymbol{p}$.

5. From each point, $n-1$ chords can be drawn (but a chord cannot be drawn from a point to itself.) There are *n* points for this to happen, but the total must be divided by 2, since chord AB is the same as chord BA. This expression comes out to be $\mathbf{\frac{n^2-n}{2}}$.

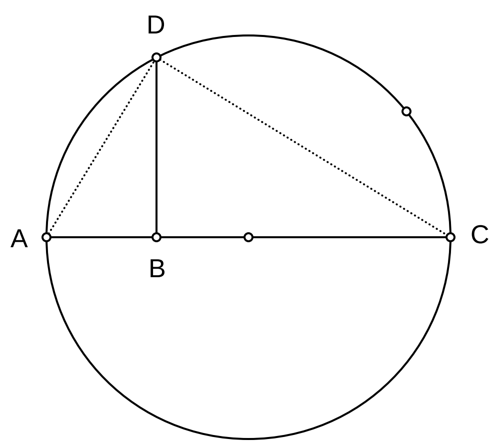
6. For ABC to exist as a triangle, the length of AC cannot exceed the sum of the lengths of AB and BC, and length of BC cannot exceed the sum of the lengths of AB and AC. So

$AC < 31m$

$AC > 11m$

$\mathbf{11m < AC < 31m}$

7.



ΔADC is a right triangle since it is inscribed in a semicircle. If a perpendicular is dropped from the right angle to the hypotenuse, then the square of its length is the product of the 2 segments it cuts off on the hypotenuse.

$(DB)^2 = AB \cdot BC$

$DB = \sqrt{9 \cdot 16} = \mathbf{12}$

8. If two similar objects are given, and if the ratio of corresponding lengths is *a*:*b*, then the ratio of corresponding areas is $a^2{:}b^2$ and the ratio of the volumes is $a^3{:}b^3$. Let the parallel base cut off a pyramid smaller and similar to the original one.

$$\frac{A\ of\ Base}{56.25} = \frac{16^2}{10^2}$$

$A\,of\ Base = \mathbf{144}$

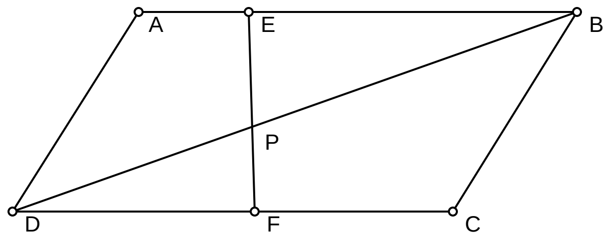
9. If $s =$ the length of one side of one of the squares,

$(2s)^2 + s^2 = (AB)^2 = 10^2$

$5s^2 = 100$

Area of the cross $= 6s^2 =$ **120**

10.



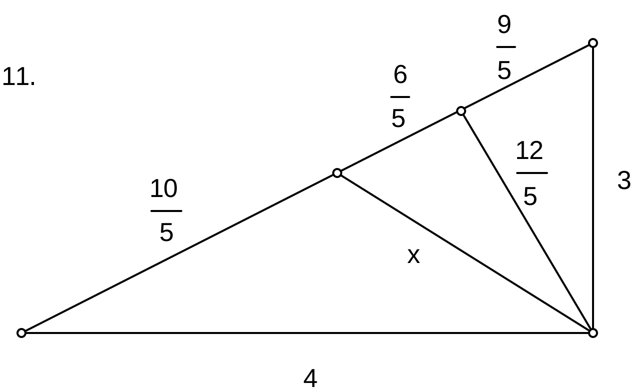
$\angle DPF = \angle BPE$

$\Delta DPF \cong \Delta BPE$

let $AB = DC = x$

$$\frac{DP}{BP} = \frac{DF}{BE} = \frac{\frac{1}{2}x}{\frac{3}{4}x} = \mathbf{\frac{2}{3}}$$

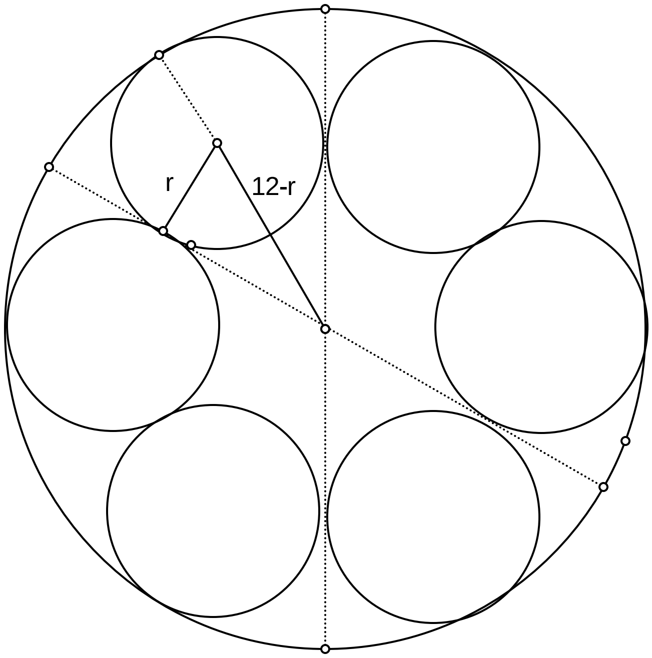
11.



$x^2 = (\frac{6}{5})^2 + (\frac{12}{5})^2$

$x = \mathbf{\frac{6}{5}\sqrt{5}}$

12.



A right triangle is made when a segment is drawn to the point of tangency of 2 of the smaller circles, the radius of a circle is drawn to that same point, and a segment is drawn from the center of the big circle to the center of the small circle. The angle opposite the side of length r is 30° (half the central angle cut off by the dashed lines), so

$2r = 12 - r$

$r = \mathbf{4}$

13.

$h = \frac{r}{5}$

$SA = 2pr^2 + 2prh$

$V = pr^2 h$

$2pr^2 + 2prh = pr^2 h$

$2(r + \frac{r}{5}) = r(\frac{r}{5})$

$2(5r + r) = r^2$

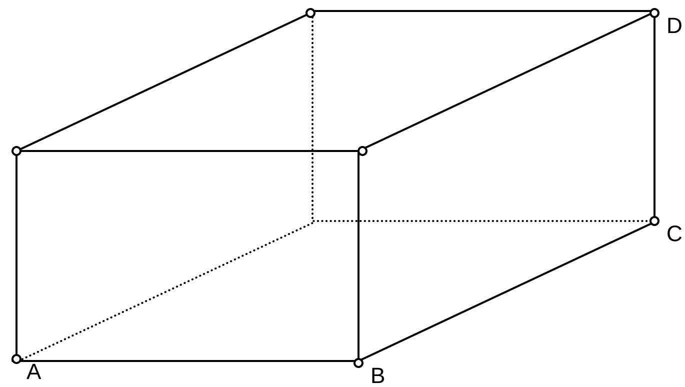
$r^2 - 12r = 0$

$r = 12$

$V = pr^2 h$

$= pr^2 \cdot \frac{r}{5} = \mathbf{\frac{1728p}{5}}$

14.



Pretend the broomstick is as long as *AD*.

$AB = 4,\ BC = 5,\ CD = 2$

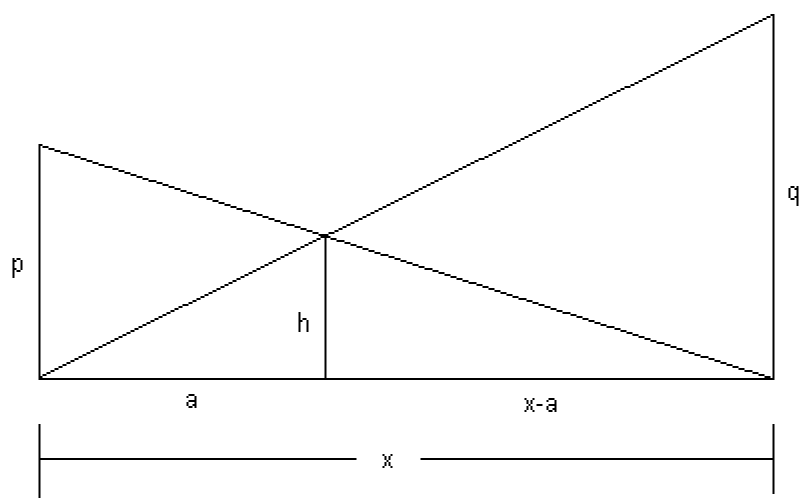
$AC^2 = AB^2 + BC^2$

$AD^2 = AC^2 + CD^2$

$= AB^2 + BC^2 + CD^2$

$AD = \sqrt{4^2 + 5^2 + 2^2} = \mathbf{3\sqrt{5}}$

15.



This problem just involves the similarity of the two pairs of triangles involved.

$$\frac{h}{p} = \frac{x-a}{x} = 1 - \frac{a}{x}$$

$$\frac{a}{h} = \frac{x}{q}$$

$$\frac{a}{x} = \frac{h}{q}$$

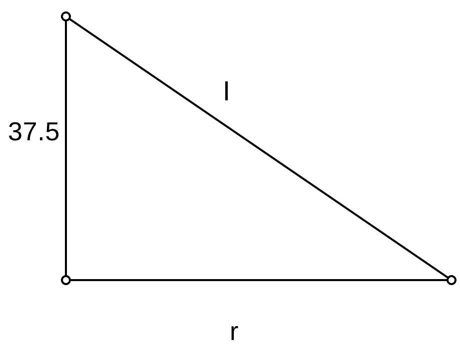
$$\frac{h}{p} = 1 - \frac{a}{x} = 1 - \frac{h}{q}$$

$$\frac{h}{p} + \frac{h}{q} = 1$$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{h}$$

$$h = \frac{\mathbf{qp}}{\mathbf{q+p}}$$

16.



r = radius of arena

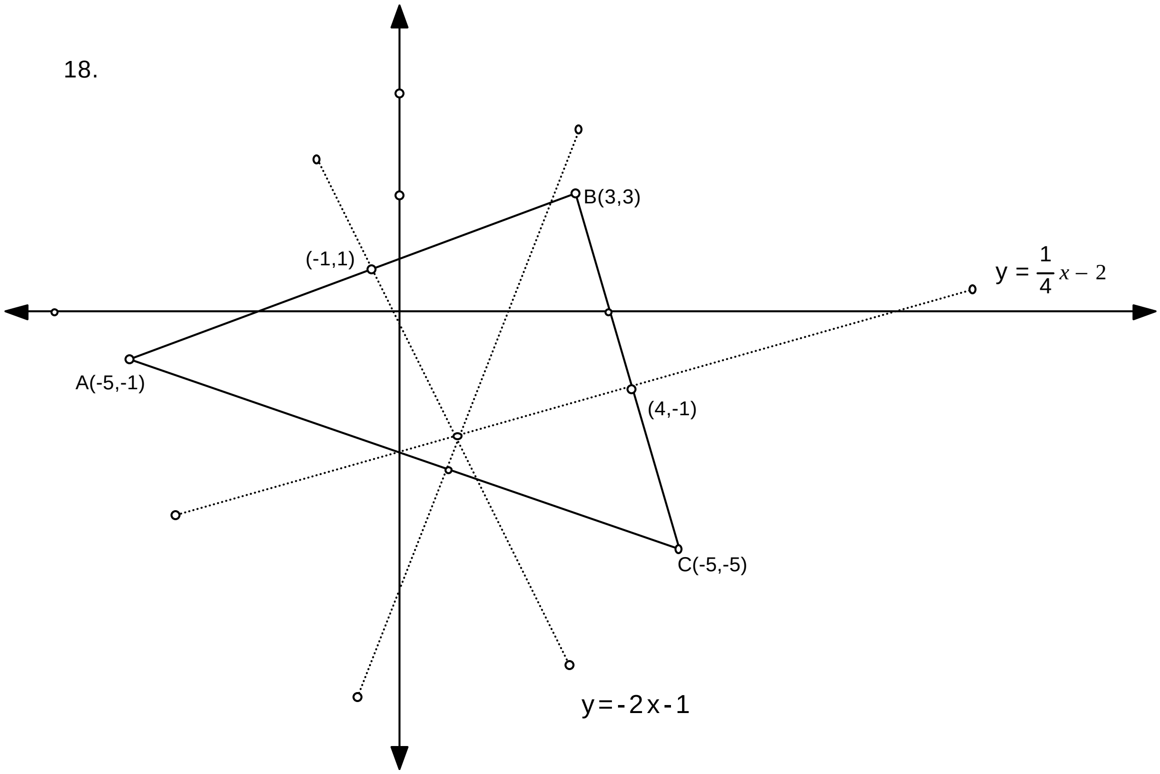
$$l^2 = r^2 + (37.5)^2$$

$$r^2 = \frac{A}{p} = 277.78$$

$$l = \sqrt{277.78 + (37.5)^2} = \mathbf{41.0}$$

17. Diagonals of the same rectangle are congruent, so $MG = ON = \mathbf{r}$.

18.



slope of $AB = \frac{1}{2}$

slope of $BC = -4$

Perpendicular bisectors of AB and BC are drawn with equations labelled. Finding where lines intersect:

$-2x - 1 = \frac{1}{4}x - 2$

$x = \frac{4}{9}$

$y = (\frac{1}{4})(\frac{4}{9}) - 2 = -\frac{17}{9}$

$x + y = \mathbf{-\frac{13}{9}}$

19. Let $4x =$ the length of the external segment

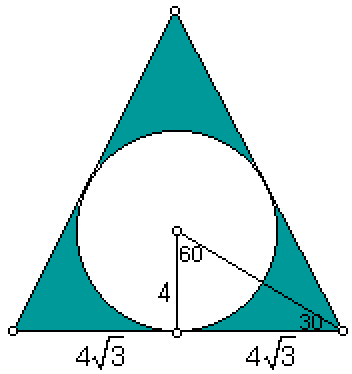
$14^2 = x(x + 3x)$

$196 = 4x^2$

$x = 7$

$4x =$ **28**

20.



$$A = \frac{\left(8\sqrt{3}\right)^2 \sqrt{3}}{4} - \left(4^2\right)p$$

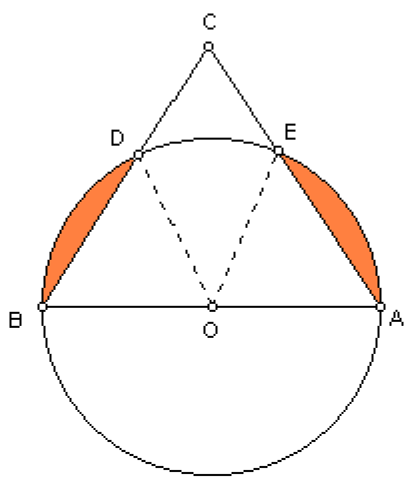
$$= 48\sqrt{3} - 16p$$

$x = 48$

$y = 16$

$x + y =$ **64**

21.



$\angle BEA = \frac{\overset{\frown}{AB} - \overset{\frown}{DE}}{2}$

$\overset{\frown}{DE} = \overset{\frown}{AB} - 2\angle BEA$

$= 180 - 2(60)$

$= 60$

$\angle BOD = \angle DOE = \angle EOA = 60$

$Area = 2(\frac{p \cdot 5^2}{6} - \frac{5^2 \cdot \sqrt{3}}{4})$

$=$ **4.529**

22. $V = B \cdot h$

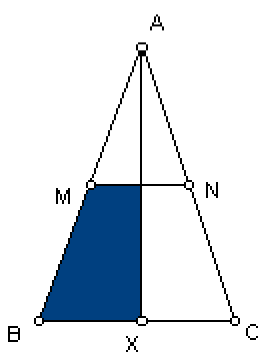
To find the area of a hexagon, consider it to be composed of 6 equilateral triangles.

$B = 6(\frac{4^2\sqrt{3}}{4}) = 24\sqrt{3}$

$V = (24\sqrt{3})(5)$

$= \mathbf{120\sqrt{3}}$

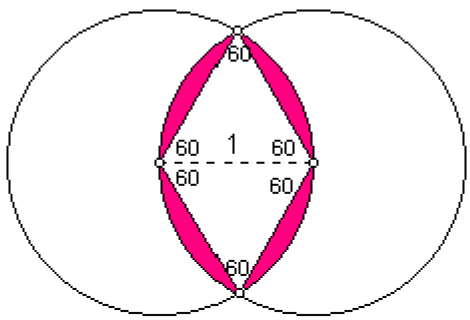
23.



MN cuts off ΔAMN, which is $\frac{1}{4}$ the area of ΔABC. This means that $MNCB$ is $\frac{3}{4}$ of ΔABC, and with the shaded region being $\frac{1}{2}$ of $MNCB$,

$\frac{\text{shaded region}}{\Delta ABC} = \mathbf{\frac{3}{8}}$

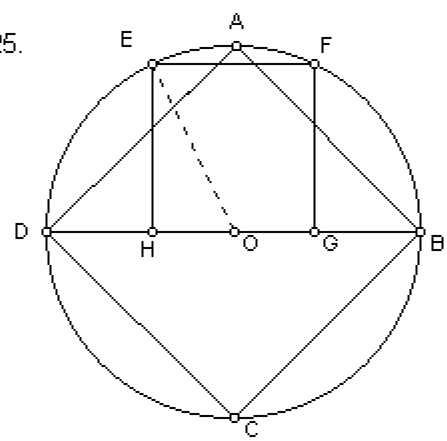
24.



$$A = 4\left(\frac{p}{6} - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{\mathbf{2}p - \mathbf{3}\sqrt{\mathbf{3}}}{\mathbf{3}}$$

25.



$CD = \sqrt{15}$

$OD = OE = \sqrt{\frac{15}{2}}$

let $x = OH$

$EH = 2(OH)$

$x^2 + (2x)^2 = \left(\sqrt{\frac{15}{2}}\right)^2$

$x^2 = \frac{3}{2}$

$A = (EH)^2 = (2x)^2 = \mathbf{6}$

26. $V = bh$

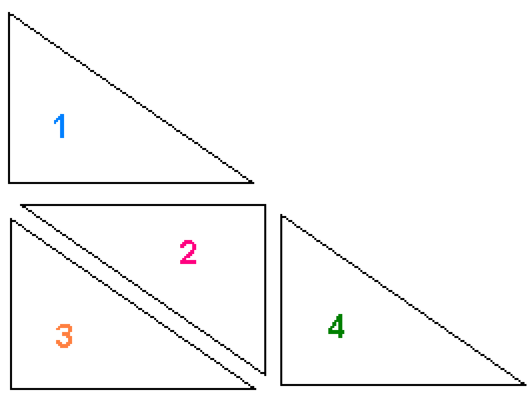
$h = 3\sqrt{2}$

A lateral edge is the hypotenuse of a 45°-45°-90° with one of its legs being the altitude.

$l = h\sqrt{2}$

$= \mathbf{6}$

27.



Let $\frac{a}{b} = \frac{\text{the lengths of the rep - tile}}{\text{the lengths of the original triangle}}$

The ratio of $\frac{a}{b}$ in this problem can only take on integer values. $\frac{a}{b} = 1$ can't be used to make a larger triangle, but instead, it implies that just the original triangle will be used.

The next possibility is $\frac{a}{b} = 2$, and then the ratio of the areas of the triangles becomes

$\frac{a^2}{b^2} = 4$, and this means 4 copies of the original triangle must be used. The diagram above shows how the **4** triangles can be arranged, which confirms the solution.

28. Inscribe a rhombus in a rectangle. Its length and width will be x and y. The area of the rectangle is xy. By drawing the diagonals of the rhombus as it is inscribed in the rectangle, it is easy to see that the rhobmus is half the area of the rectangle, or $\mathbf{\frac{xy}{2}}$.

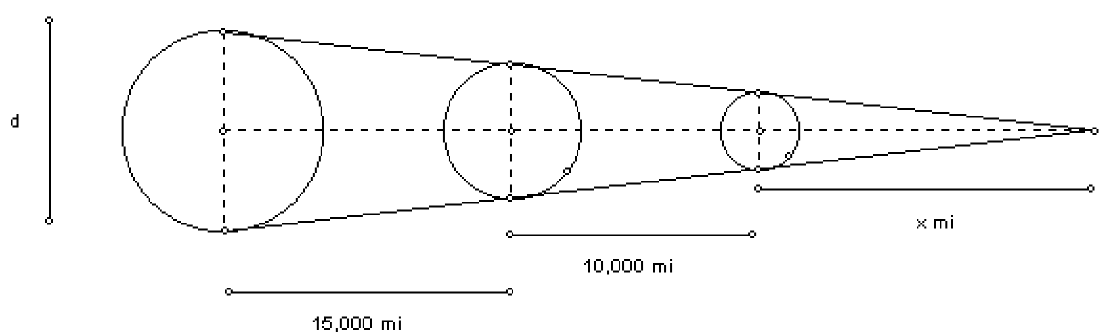
29. Given a triangle of any length sides a, b, and c, let s(semiperimeter) = $\frac{a+b+c}{2}$.

Heron's Formula states that the area of a triangle is found by

$A = \sqrt{s(s-a)(s-b)(s-c)}$

So $A = \sqrt{14 \cdot 9 \cdot 4 \cdot 1} = \mathbf{6\sqrt{14}}$

30.



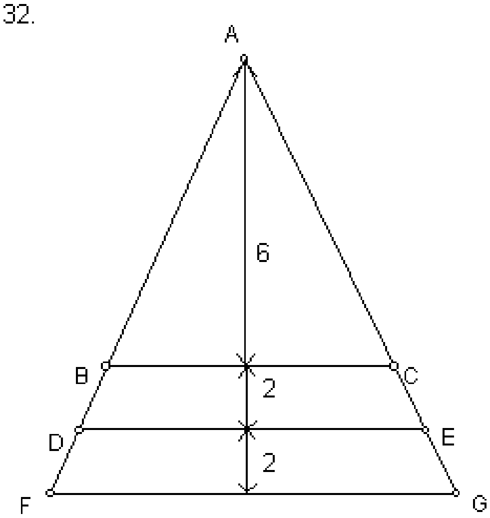
From the point that is x mi away from the samllest planet, 2 lines extend through the endpoints of the diameters of all of the palnets. This creats a large triangle, and the diagmeters of the smaller 2 planets cut off and create smaller triangles similar to the big one. With corresponding parts of similar triangles being similar, proportions can be used to find x, and then d.

$$\frac{x}{3{,}000} = \frac{x = 10{,}000}{3{,}000}$$
$$x = 6{,}000$$
$$\frac{6{,}000}{3{,}000} = \frac{6{,}000 + 10{,}000 + 15{,}000}{d}$$
d = **15,500**

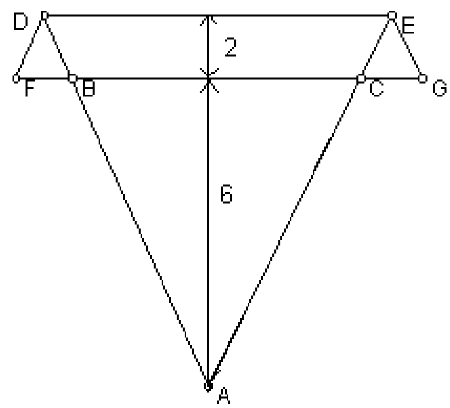
31. The area of a triangle can be expressed as $A = \frac{abc}{4R}$, where $R =$ the circum-radius. Since the triangle is isoceles, the median to the side of length 12 is also an altitude. The altitude is of length 8, so the area is 48.

$$48 = \frac{10 \cdot 10 \cdot 12}{4R}$$
$R =$ **6.25**

32.



not folded



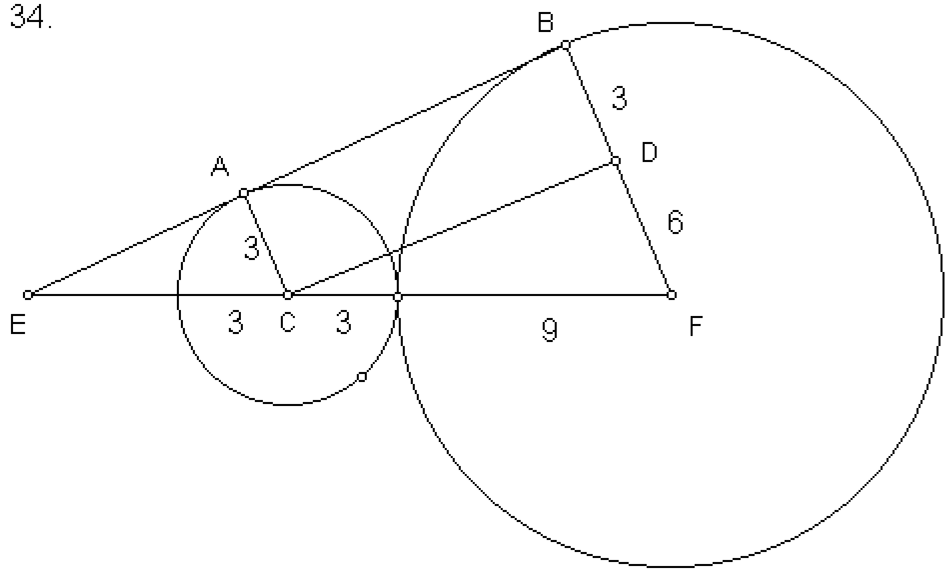
folded

The above diagram assumes the altitude from A is 10, but the length is not important. However, since the ratio of the area of triangles is $\frac{a^2}{b^2}$ when their lengths are in ratio $\frac{a}{b}$, the length from A to BC should be $\sqrt{0.36} = \frac{6}{10}$ of the length from A to FG. Also, since the length from BC to DE is the same as from DE to FG (as in *folded* diagram), those lengths should both be $\frac{2}{10}$ of the original altitude. $FG = 12$, and DE is the crease, so the similar triangles show DE = **9.6**.

33. The only way to draw two unequal diagonals in a regular hexagon is to draw a 30°-60°-90° right triangle, with the diagonals being the sides oppose the 60° and 90° angles. Let k = length of one side of the hexagon.

$$\frac{A \text{ of right triangle}}{A \text{ of hexagon}} = \frac{\frac{k \cdot k\sqrt{3}}{2}}{6 \cdot \frac{k^2\sqrt{3}}{4}} = \mathbf{\frac{1}{3}}.$$

34.



C and F are the centers of the circles, and BF is a radius of circle F. CD is drawn so that $CD \perp BF$, making $ABDC$ a rectangle. $AB = CD = 3$, $DF = 6$, and

$AB = CD = \sqrt{CF^2 - DF^2} = \mathbf{6\sqrt{3}}$.

35. $\frac{V_b}{V_a} = (\frac{2}{1})^3 = \mathbf{8}$. (see question 8)

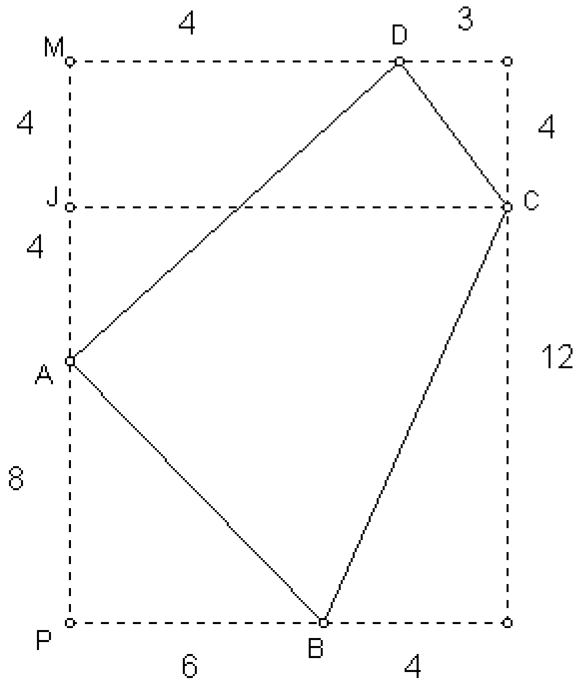
36. Justlike the area of a trapezoid being $h \cdot \frac{b_1 + b_2}{2}$, the volume of this cylinder is calculated as $h \cdot \frac{h_1 + h_2}{2}$. (To understand why, imagine slicing the cylinder 15 cm from the base. Take the sliced off solid and rotate it 180° about its only straight edge. Now a "normal" right circular cylinder is formed with an equal volume to the original cylinder, and the new cylinder's height is just the average of the original one's maximum and minimum heights.)

$V = 4^2 p(\frac{12+18}{2}) = \mathbf{240}\boldsymbol{p}$

37. Draw a right circular cone whose base is the cross-section and whose slant height is r and altitude is k. The radius of the base is $\sqrt{r^2 - k^2}$ b the Pythagorean Theorem, so its area is

$A = p(\sqrt{r^2 - k^2})^2 = \boldsymbol{p}(\mathbf{r}^2 - \mathbf{k}^2)$

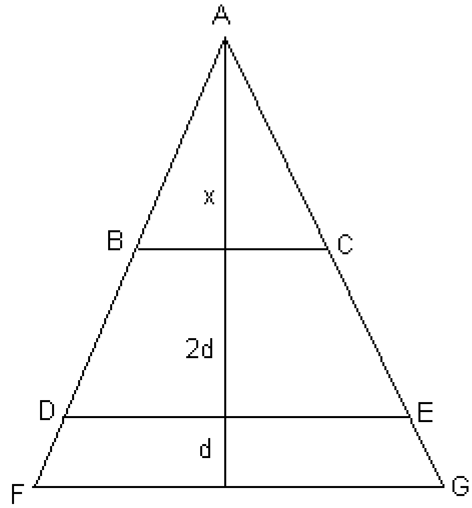
38.



38. The area of the shape is the same as the area left when the areas of non-included triangles are subtracted from the area of the circumscribed rectangle.

$A = 160 - 28 - 6 - 24 - 24 = \mathbf{78}$.

39.



$BCGF$ is the original trapezoid.

$BC = 7, \quad FG = 13$

With two unknowns, x and d, x needs to be solved for in terms of d.

$$\frac{x}{7} = \frac{x+3d}{13}$$

$$x = \frac{7}{2}d$$

Now to find an expression for DE

$$\frac{\frac{11}{2}d}{DE} = \frac{\frac{7}{2}d}{7}$$

$$DE = 11$$

Solving for d

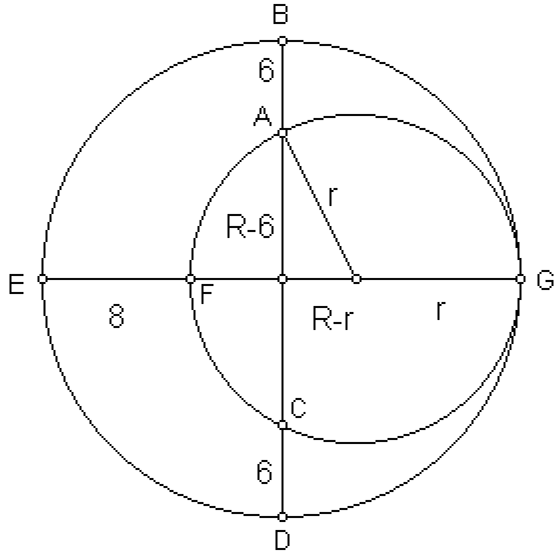
$$16 = \frac{(13)(\frac{13}{2}d)}{2} - \frac{(11)(\frac{11}{2}d)}{2}$$

$$d = \frac{4}{3}$$

$$A = \frac{(11)(\frac{11}{2}d)}{2} - \frac{(7)(\frac{7}{2}d)}{2} = 24$$

This not offered, the correct choice is **none of these (E)**.

40.



Two relationships between R and r are needed to solve for either variable.

$EG = 2R$

$EG = EF + FG = 8 + 2r$

$2R = 8 + 2r$

$R = r + 4$

Drawing the radius from the center of the smaller circle to point A reveals

$(R - r)^2 + (R - 6)^2 = r^2$

Now we have the two equations. Substituting what we have from the first equation,

$(4)^2 + (r - 2)^2 = r^2$

$r = 5$

$R = r + 4 = 9$

$R + r =$ **14**