Finite Differences and Recurrence Relations

Solutions.

1. Make an argument to show that the third differences for cubic data is actually 6a.

X	Y	△1	△2	△3
x - 2	$a(x-2)^3+b(x-2)^2$ $+c(x-1)+d$			
		$a\left(3x^2+9x+7\right)+$ $b\left(2x+3\right)+c$		
x - 1	$a(x-1)^3+b(x-1)^2$ $+c(x-1)+d$		$a\left(6x+6\right)+2b$	
		$a\left(3x^2+3x+1\right)+$ $b\left(2x+1\right)+c$		$6a$
x	$a(x)^3+b(x)^2$ $+cx+d$		$a\left(6x\right)+2b$	
		$a\left(3x^2-3x+1\right)+$ $b\left(2x-1\right)+c$		$6a$
x + 1	$a(x+1)^3+b(x+1)^2$ $+c(x+1)+d$		$a\left(6x-6\right)+2b$	
		$a\left(3x^2-9x+7\right)+$ $b\left(2x-3\right)+c$		
x + 2	$a(x+2)^3+b(x+2)^2$ $+c(x+2)+d$			

2. Find the sum of the squares of the first 100 positive integers.

Of course this is arithmetic, so we all know the quick formula for this, that is $S_{100}=\left(1+100\right)\frac{100}{2}=(101)(50)=5050$, but let's do it with the finite differences.

X	1		2		3		4		5
Y	1		3		6		10		15
△1		2		3		4		5	
△2			1		1		1		

It looks like the second finite differences are all one, so the function for the partial sums is quadratic with leading coefficient ½. So we get $S(n)=\frac{1}{2}n^2+bn+c$ and see that $S(1)=\frac{1}{2}(1)^2+b+c=1$ and $S(2)=\frac{1}{2}(2)^2+2b+c=3$, so $b+c=\frac{1}{2}$, and $2b+c=1$. Solving this system of equations gives us $b=\frac{1}{2},c=0$, so $S(n)=\frac{1}{2}n^2+\frac{1}{2}n=\frac{1}{2}\cdot n(n+1)$.

3. Find the sum of the cubes of the first 50 positive integers.

X	1		2		3		4		5		6		7
Y	1		9		36		100		225		441		784
△1		8		27		64		125		216		343	
△2			19		37		61		91		127		
△3				18		24		30		36			
△4					6		6		6				

Since the fourth differrences are constant, we know that this is a fourth-degree polynomial of the form $S(n)=\frac{6}{4!}n^4+bn^3+cn^2+dn+f$. There are several approaches to finding the remaining coefficients. The most efficient way may be to notice that the desired sums are all perfect squares, so $S(n)=\left(\frac{1}{2}n^2+bn+c\right)^2$,

and $S(1)=\left(\frac{1}{2}\cdot 1^2+b+c\right)^2=1\Rightarrow b+c=\frac{1}{2}$ and

$S(2)=\left(\frac{1}{2}\cdot 2^2+2b+c\right)^2=9\Rightarrow 2+2b+c=3$. This system yields $b=\frac{1}{2},c=0$, so our desired sum is $S(n)=\left(\frac{1}{2}n^2+\frac{1}{2}n\right)^2=\frac{1}{4}n^2(n+1)^2$,

4. Find the sum $1-4+9-16+\cdots-100^2$.

Notice that the partial sums for the triangular numbers with the added fact that every other one is negative. So if we find the formula for the triangular numbers, we are nearly done.

n	1	2	3	4	5	6
S(n)	1	-3	6	-10	15	-21

n	1		2		3		4		5
S(n)	1		3		6		10		15
△1		2		3		4		5	
△2			1		1		1		

So we see that the sum is quadratic of the form $S(n) = \left(\frac{1}{2}\cdot n^2 + bn + c\right)$.

Substituting we find that $S(1) = \left(\frac{1}{2}\cdot 1^2 + b + c\right) = 1$ and $S(2) = \left(\frac{1}{2}\cdot 2^2 + 2b + c\right) = 3$,

So $b + c = \frac{1}{2}, 2b + c = 1 \Rightarrow b = \frac{1}{2}, c = 0$. So the desired sum is

$$S(n) = (-1)^{n+1}\left(\frac{1}{2}\right)\left(n^2 + n\right).$$

5. The Lucas sequence begins with 2, then 1, and then the remaining terms are found as in the Fibonacci sequence. Write the recursive and closed forms for the terms of this sequence.

The recursive form for the Lucas sequence is simply $t_0 = 2, t_1 = 1, t_n = t_{n-1} + t_{n-2}, n \geq 2$, so the terms are 2, 1, 3, 4, 7, 11, 18, 29, 47, …. To find the closed formula, rewrite the recursive equation as $t_n - t_{n-1} - t_{n-2} = 0$ and notice that this is exactly like the Fibonacci solution above, so

$t_n = c_1 \cdot r_1^n + c_2 \cdot r_2^n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$, but since the initial values are

different, we have $t_0 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = c_1 + c_2 = 2$ and

$$t_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1 = c_1 + c_2 = 1$$

6. The first term of a sequence is 0 and the second is 1. For all $n \geq 2, t_n = 3 \cdot t_{n-1} - 2 \cdot t_{n-2}$. Find the first 5 terms of the sequence, the recursive, and the closed form for the terms of this sequence.

The first five terms are 0, 1, 3, 7, and 15, so it looks like the function is simply $t_n = 2^n - 1$. Let's use the method above to rewrite the recursive equation as $t_n - 3 \cdot t_{n-1} + 2 \cdot t_{n-2} = 0$ and proceed. We will have $cr^n - 3cr^{n-1} + 2cr^{n-2} = 0$, so $cr^{n-2}\left(r^2 - 3r + 2\right) = 0$. The solutions to the second factor are $r_1 = 1,\ r_2 = 2$. Now we have $t_n = c_1 \cdot r_1^n + c_2 \cdot r_2^n = c_1\left(1\right)^n + c_2\left(2\right)^n$. Now take the first two given values to find the values for c_1, c_2. We get the system of equations:

$$\begin{array}{l} 0 = c_1\left(1\right)^0 + c_2\left(2\right)^0 = c_1 + c_2 \Rightarrow c_2 = -c_1 \\ 1 = c_1\left(1\right)^1 - c_1\left(2\right)^1 = c_1 - 2c_2 \Rightarrow -1 = c_1, \Rightarrow c_2 = 1 \end{array}.$$

This means then that $t_n = -1\left(\left(1\right)^n - \left(2\right)^n\right) = 2^n - 1$.

7. Sequence $\left(a_1, a_2, a_3, \ldots\right)$ is defined recursively by $a_1 = 0, a_2 = 100$ and $a_n = 2a_{n-1} - a_{n-2} - 3$. Find the greatest term in the sequence $\left(a_1, a_2, a_3, \ldots\right)$. {Duke Math Meet 1998 Team Problem #9)

Since $a_n - 2a_{n-1} + a_{n-2} = -3$, this is a non-homogenous recurrence relation, so the above method does not work. However, finite difference will work. First, generate several values:

n	0		1		2		3		4
f(n)	0		100		197		291		382
△1		100		97		94		91	
△2			-3		-3		-3		

We see now that the function is quadratic, and in the form $f(n) = -\frac{3}{2}n^2 + bn + c$. We also know that $f(0) = 0 \Rightarrow c = 0$. Similarly we also know that $f(1) = 100 = -\frac{3}{2}(1)^2 + b(1) \Rightarrow b = 101.5$. So we have $f(n) = -\frac{3}{2}n^2 + 101.5n$ and the maximum value of this function occurs when $x = -\frac{b}{2a} = \frac{101.5}{3} = 33.8\bar{3}$. Since our function is a sequence (whose domain is the Whole Numbers), we need to check the values 33 and 34 to find the maximum. Since 34 is closer to the real-valued vertex, it yields 1717, which is the maximum value.

8. Find k given that $\begin{cases} f(0) = k \\ f(n) = f(n+1) - 3n - 2 \end{cases}$ and $f(-50) = 4000$.

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This problem can be done, among other ways, by using finite differences. First, it might help to rewrite the recursive equation as $f(n+1) = f(n) + 3n + 2$. Now generate a table of the first several values.

x	0		1		2		3		4
y	k		k + 2		k + 7		k + 15		k + 26
△1		2		5		8		11	
△2			3		3		3		

We see now that the function is quadratic, and in the form $f(n) = \frac{3}{2}n^2 + bn + k$.

We also know that $f(1) = k + 2 = \frac{3}{2} \cdot 1^2 + b \cdot 1 + k \Rightarrow b = \frac{1}{2}$. Finally we are told that $f(-50) = 4000 = \frac{3}{2}\left(-50\right)^2 + (-50) + k \Rightarrow k = 275.$