Triangles

Solutions:

1. Two sides of a triangle are 7 and 9 while the median to the third side has length 5. Find the length of the third side.

   The median theorem tells us that $m_c = \sqrt{\frac{a^2+b^2}{2} - \frac{c^2}{4}}$, so

   $5 = \sqrt{\frac{7^2+9^2}{2} - \frac{c^2}{4}} \Rightarrow 25 = \frac{49+81}{2} - \frac{c^2}{4} \Leftrightarrow 100 = 2(130) - c^2$, so

   $c^2 = 160 \Rightarrow c = 4\sqrt{10}$

2. The sides of a triangle are consecutive integers and the area is an integer. Find the triangle with the smallest perimeter that is not a right triangle. Are there others?

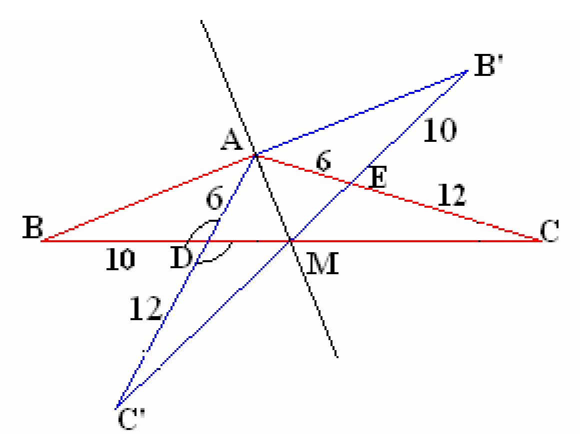
   Let the sides be $b-1, b, b+1$, so the area is

   $K = \sqrt{\frac{3b}{2}\left(\frac{3b}{2}-(b-1)\right)\left(\frac{3b}{2}-(b)\right)\left(\frac{3b}{2}-(b+1)\right)} = \sqrt{\frac{3b}{2}\left(\frac{b+2}{2}\right)\left(\frac{b}{2}\right)\left(\frac{b-2}{2}\right)}$. This

   simplifies to $\frac{b}{4}\sqrt{3(b-2)(b+2)}$, so it looks like b has to be a multiple of 4 or one of $b-2$ or $b+2$ must be a perfect square or 3 times a perfect square. If $b=4$, then the area is an integer, (6), but the triangle is a 3-4-5 right triangle. The next value that works, is $b=14$ with the area

   $\frac{14}{4}\sqrt{3(14-2)(14+2)} = \frac{14}{4}\sqrt{3(12)(16)} = \frac{14}{4}(6)(4) = 84$. There are others. The next two are 193,194,195 and 2710,2702,2703 with area 16296 and 316340.

3. Triangle ABC is reflected in (or about) its median $\overline{AM}$ (extended) as shown. If $AE=6,\ EC=12,\ BD=10$ and $AB = k\sqrt{3}$, compute *k*. 1987 ARML I8

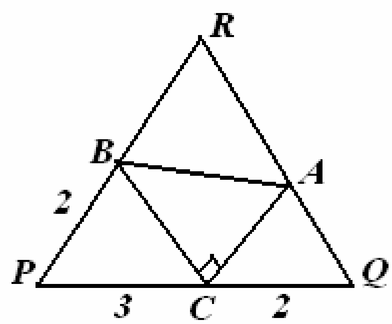


The median AM divides triangle ABC into two triangles, ABM and CAM which both have the same area. Since B'AM is the reflection of BAM, and C'AM is the reflection of CAM, all of these triangles have the same area. Now look at triangles BAD and C'DM. They have the same area as well (take triangle ADM off of BAM and C'AM). the areas can be found using the formula $\frac{1}{2}BD \cdot DA \cdot \sin(\angle ADB) = \frac{1}{2}C'D \cdot DM \cdot \sin\left(\angle MDC'\right)$, but since the angles are equal, we have it has the same area $BD \cdot DA \cdot = C'D \cdot DM$, so $6 \cdot 10 = AD \cdot 12 \Rightarrow AD = 5$. So EM = 5 also. This makes CM = 15. Now the law of cosines gives $\cos\left(\angle C\right) = \frac{12^2 + 15^2 - 5^2}{2(12)(15)} = \frac{344}{360} = \frac{43}{45}$. Now in triangle ABC,

$AB^2 = 30^2 + 18^2 - 2(30)(18)\frac{43}{45} = 900 + 324 - 2(2)(6)43 = 192$, so

$AB = \sqrt{192} = 8\sqrt{3}$.

4. Right triangle ABC (hypotenuse $\overline{AB}$) is inscribed in equilateral triangle PQR, as shown. If $PC = 3$, and $BP = CQ = 2$, compute AQ. ARML 1991 I7



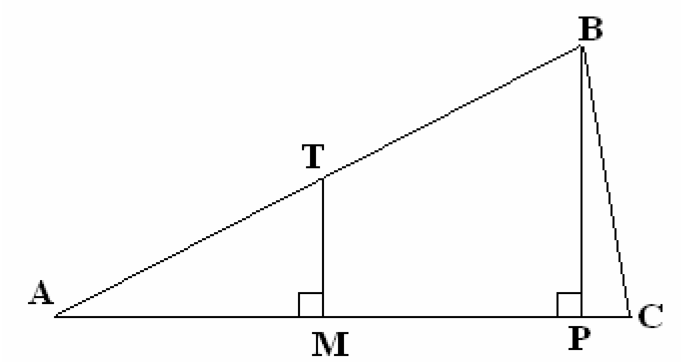
Using the law of cosines we can find BC, AC, and AB and then use the Pythagorean Theorem to relate these. First let $AQ = x, AR = 5 - x$ and note that $BR = 3$. So, $BC^2 = 2^2 + 3^2 - 2(2)(3)\left(\frac{1}{2}\right) = 4 + 9 - 6 = 7$,

$AC^2 = 2^2 + x^2 - 2(2)(x)\left(\frac{1}{2}\right) = x^2 - 2x + 4$, and

$AB^2 = 3^2 + (5-x)^2 - 2(3)(5-x)\left(\frac{1}{2}\right) = x^2 - 7x + 19$. So now we have

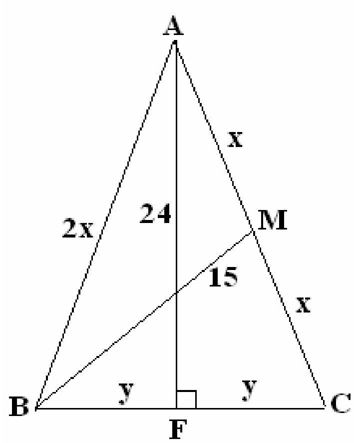
$x^2 - 7x + 19 = \left(x^2 - 2x + 4\right) + \left(7\right)$, so $5x = 8 \Rightarrow x = \frac{8}{5} = \boxed{1\frac{3}{5}}$.

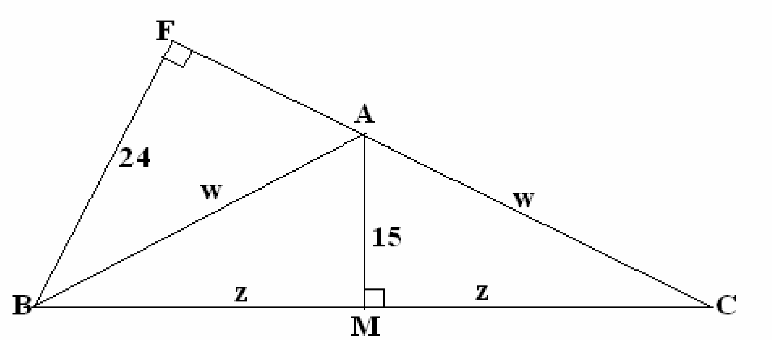
5. In triangle ABC, the perpendicular bisector of $\overline{AC}$ intersects $\overline{AC}$ at M and $\overline{AB}$ at T. If the area of triangle AMT is $\frac{1}{4}$ the area of triangle ABC, and $\measuredangle A + \measuredangle C = 128^{\circ}$, compute the number of degrees in angle A. ARML 1988 I4



Since the area of AMT is one-fourth the area of ABC, the altitude MT must be half the altitude BP. Now, since MT is half of BP, AT = TB and AM = MP, but we know that AM = MC, so C must coincide with P, making triangle ABC a right triangle and since $m\angle A + m\angle C = 128^o, m\angle A = \boxed{38^o}$

6. An isosceles triangle has a median equal to 15 and an altitude equal to 24. This information determines exactly two triangles. Compute the area of each of these triangles. ARML 1994 T4





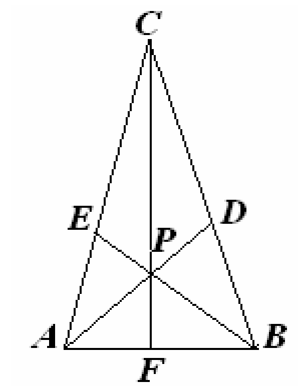
The two triangles are shown above, with BM the median with length 15 in the first and AF the altitude with length 24. In the first triangle, we know that

$15 = \sqrt{\frac{(2x)^2 + (2y)^2}{2} - \frac{(2x)^2}{4}} = \frac{1}{2}\sqrt{4x^2 + 8y^2}$ and $(2x)^2 = 24^2 + y^2$, so

$900 = 4x^2 + 8y^2$ and $576 = 4x^2 - y^2$, which makes

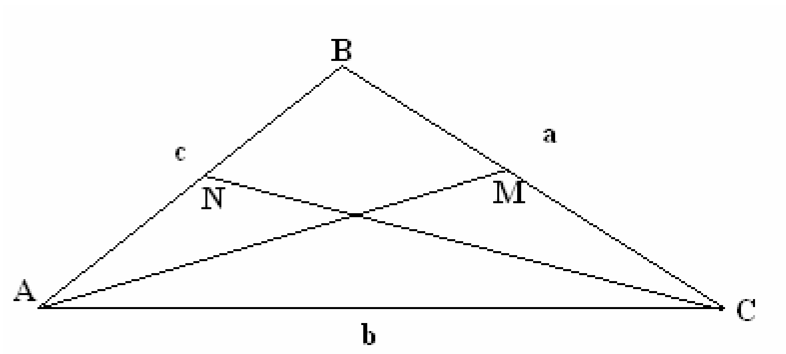
$900 - 8y^2 = 576 + y^2 \Leftrightarrow 9y^2 = 324 \Rightarrow y^2 = 36 \Rightarrow y = 6$ making the area 144. In the second triangle we have $w^2 = 15^2 + z^2$ and $\frac{w}{15} = \frac{2z}{24} \Rightarrow 24w = 30z \Rightarrow w = \frac{5}{4}z$, so $\left(\frac{5}{4}z\right)^2 = 15^2 + z^2 \Rightarrow \frac{25}{16}z^2 = 15^2 + z^2 \Rightarrow \frac{9}{16}z^2 = 15^2 \Rightarrow z = 20$ and $w = 25$, which makes $AF = 7$. Thus the area of this triangle is 300.

7. Point P is inside $\triangle ABC$. Line segments $\overline{APD}$, $\overline{BPE}$, and $\overline{CPE}$ are drawn with D on $\overline{BC}$, E on $\overline{CA}$, and F on $\overline{AB}$. (See figure). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$. AIME 1989 #15



8. The points (0,0), (a,11), and (b,37) are the vertices of an equilateral triangle. Find the value of ab. AIME 1994 # 8

9. A direct proof establishes that equal sides implies equal medians. If $\overline{AB} \cong \overline{BC}$, then $\overline{AN} \cong \overline{CM}$, and since $\angle MAC \cong \angle NCA$ and $\overline{AC} \cong \overline{AC}$, we have $\triangle MAC \cong \triangle NCA$, so $\overline{AM} \cong \overline{CN}$. To prove that equal medians implies equal sides, we can't prove congruent triangles, so we will take an indirect approach. So we will assume the medians are equal, but the sides are not. So assume $m\overline{AM} = m\overline{CN}$, then



$$\sqrt{\frac{b^2+c^2}{2} - \frac{a^2}{4}} = \sqrt{\frac{b^2+a^2}{2} - \frac{c^2}{4}} \Rightarrow \frac{b^2+c^2}{2} - \frac{a^2}{4} = \frac{b^2+a^2}{2} - \frac{c^2}{4} \Rightarrow 2\left(b^2+c^2\right) - a^2 = 2\left(b^2+a^2\right) - c^2$$

, so $2b^2 + 2c^2 - a^2 = 2b^2 + 2a^2 - c^2 \Rightarrow 3a^2 = 3c^2 \Rightarrow a = c$. This is really not an indirect proof, is it?