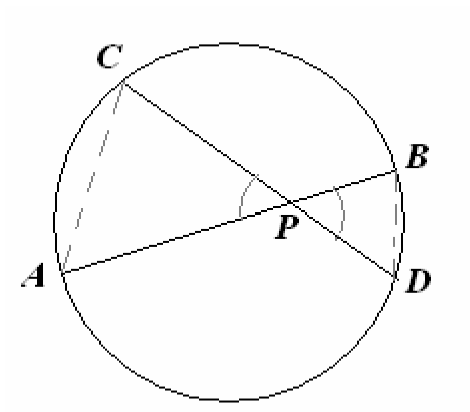
Geometry Facts – Circles & Cyclic Quadrilaterals

Circles, chords, secants and tangents combine to give us many relationships that are useful in solving problems.

Power of a Point Theorem:

The simplest of these theorems pertains to two chords of a circle that intersect in the interior of the circle.

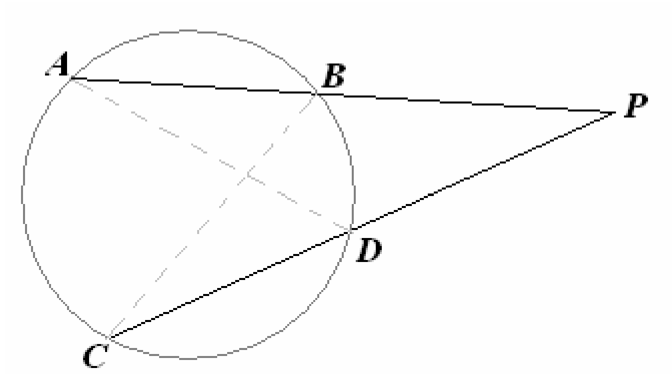


The theorem tells us that $AP \cdot PB = CP \cdot PD$. This is easy to prove since $\triangle APC \approx \triangle BPD$. We know that $\frac{AP}{CP} = \frac{PD}{PB}$.

Before going on to the other Power of a Point Theorems, it might be worth noting something about the angles formed by secants, chords, and tangents to circles. In the above figure, $\angle APC \cong \angle BPD$, since they are vertical angles, but they are both equal to $\frac{1}{2}\left(m\widehat{AC} + m\widehat{BD}\right)$.

Likewise, $\angle CDB \cong \angle CAB$, and both have measure $\frac{1}{2}m\widehat{BC}$.

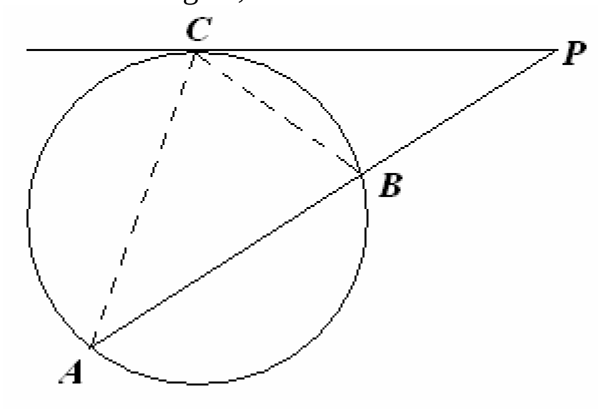
Now let's look at two secant lines.



Written & Compiled by John Goebel, NCSSM Problem Solving Course, 2006

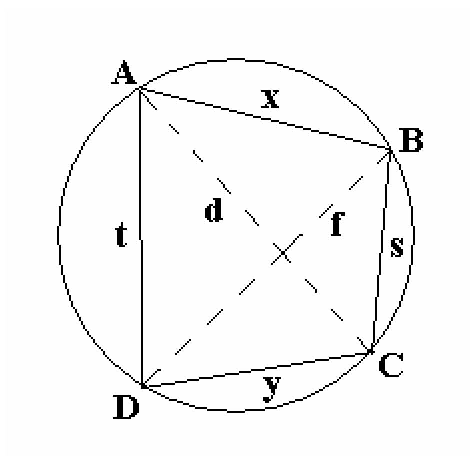
Is it easy to show that $\triangle PAD \approx \triangle PCB$, so $\frac{PB}{PC} = \frac{PD}{PA}$. When this is written as $PB \cdot PA = PD \cdot PC$, we have our second "Power of a Point" theorem. The angle $m\angle APC = \frac{1}{2}\left(m\widehat{AC} - m\widehat{BD}\right)$.

Now let's look at a secant and a tangent,



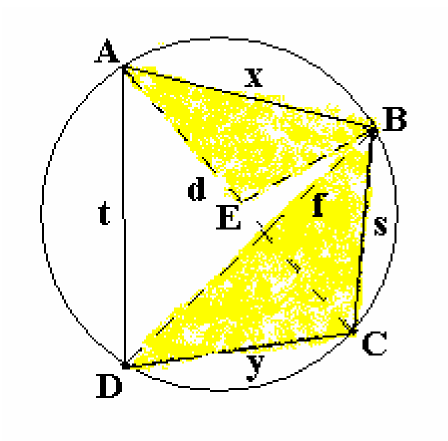
We need to know that $m\angle CAB = \frac{1}{2} m\widehat{CB}$ and $m\angle PCB = \frac{1}{2} m\widehat{CB}$, so we have $\triangle PCA \approx \triangle PBC$, and $\frac{PC}{PA} = \frac{PB}{PC}$, or $PC^2 = PA \cdot PB$. The angle at P, has measure $m\angle CPB = \frac{1}{2}\left(m\widehat{AC} - m\widehat{BC}\right)$.

Closely related to circles are the Cyclic Quadrilaterals. These are quadrilaterals that are inscribed in a circle, that is, their vertices are on a circle.

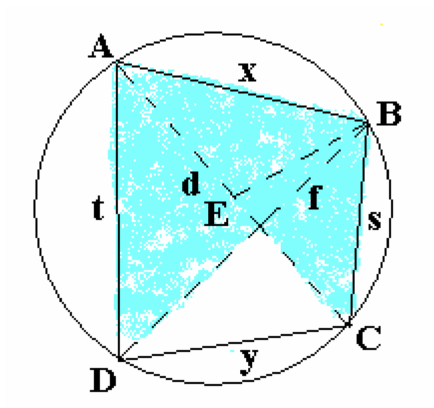


First, it should be obvious that $\angle A$ and $\angle C$ are supplementary, as are $\angle B$ and $\angle D$, since both pairs cut off opposite halves of the circle.

Ptolemy's Theorem: If $ABCD$ is a cyclic quadrilateral, then the sum of the products of opposite sides is equal to the product of the diagonals.



Proof: Drop segment BE so that $\angle ABE \cong \angle CBD$. Since $\angle BAE \cong \angle CDB$, we know that $\triangle ABE \sim \triangle DBC$. Thus $\frac{AB}{AE} = \frac{BD}{DC} \Rightarrow AB \cdot DC = AE \cdot BD$.

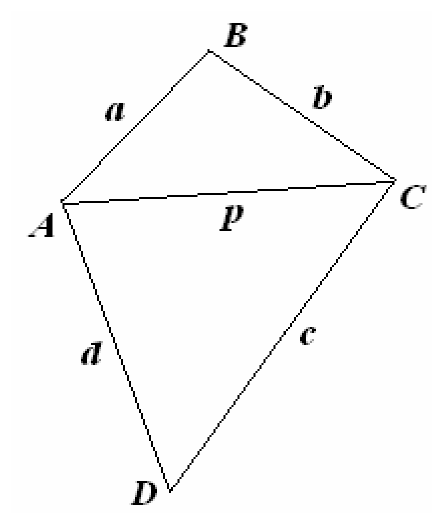


Also since $\angle ADB \cong \angle ACB$ and $\angle ABD \cong \angle CBE$ we have $\triangle ABD \sim \triangle EBC$. Thus $\frac{AD}{BD} = \frac{EC}{BC} \Rightarrow AD \cdot BC = BD \cdot EC$. Therefore

$$AB \cdot DC + AD \cdot BC = AE \cdot BD + BD \cdot EC$$
$$= (AE + EC)BD = AC \cdot BD$$

Bretschneider's Formula: In any quadrilaterial, ABCD, with sides a, b, c, and d, the area is $K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{\angle B + \angle D}{2}\right)}$, where $s = \frac{a+b+c+d}{2}$

Proof:



First draw diagonal AC and calculate the area of the quadrilateral as the sum of the areas of the two triangles formed.

$$K = \frac{1}{2}ab \cdot sin(B) + \frac{1}{2}cd \cdot sin(D)$$

Square this equation:

(1) $$K^2 = \frac{1}{4}a^2b^2\sin^2(B) + \frac{1}{2}abcd\sin(B)\sin(D) + \frac{1}{4}c^2d^2\sin^2(D).$$

Now use the law of cosines to twice as follows:

$p^2 = a^2 + b^2 - 2ab\cos(B) = c^2 + d^2 - 2cd\cos(D)$. Rewrite this last equation as $a^2 + b^2 - c^2 - d^2 = 2ab\cos(B) - 2cd\cos(D)$ and then square this equation, set equal to zero, and write as follows: $\left(2ab\cos(B) - 2cd\cos(D)\right)^2 - \left(a^2+b^2-c^2-d^2\right)^2 = 0$. Square the first term and divide through by 16 to get the following equation:

(2) $$\frac{4a^2b^2\cos^2(B) - 8abcd\cos(B)\cos(D) + 4c^2d^2\cos^2(D) - \left(a^2+b^2-c^2-d^2\right)^2}{16} = 0$$

Now add equations (1) and (2) :

$$\begin{aligned}
K^2 &= \frac{a^2b^2\sin^2(B)}{4} + \frac{abcd\sin(B)\sin(D)}{2} + \frac{c^2d^2\sin^2(D)}{4} \\
&\quad + \frac{a^2b^2\cos^2(B)}{4} - \frac{abcd\cos(B)\cos(D)}{2} + \frac{c^2d^2\cos^2(D)}{4} - \frac{\left(a^2+b^2-c^2-d^2\right)^2}{16} \\
&= \frac{a^2b^2}{4} + \frac{abcd}{2}\left(\sin(B)\sin(D) - \cos(B)\cos(D)\right) + \frac{c^2d^2}{4} - \frac{\left(a^2+b^2-c^2-d^2\right)^2}{16} \\
&= \frac{1}{4}\left(a^2b^2 + c^2d^2 - 2abcd\cos\left(B+D\right)\right) - \frac{\left(a^2+b^2-c^2-d^2\right)^2}{16} \\
&= \frac{4a^2b^2 + 4c^2d^2 - \left(a^2+b^2-c^2-d^2\right)^2}{16} - \frac{abcd\cos\left(B+D\right)}{2}
\end{aligned}$$

Square the term on the left.

$$K^2 = \frac{2a^2b^2 + 2a^2c^2 + 2a^2d^2 + 2b^2c^2 + 2b^2d^2 + 2c^2d^2 - a^4 - b^4 - c^4 - d^4}{16} - \frac{abcd\cos\left(B+D\right)}{2}$$

Now by adding and subtracting $8abcd$ to the numerator on the left, and some pretty fancy factoring, we have

$$K^2 = \frac{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}{16} - \frac{abcd}{2} - \frac{abcd\cos\left(B+D\right)}{2}.$$

Now, if $s = \dfrac{a+b+c+d}{2}$, we have

$K^2 = (s-a)(s-b)(s-c)(s-d) - \dfrac{abcd}{2}\left(1+\cos(B+D)\right)$. Now, using the half angle cosine formula, we have

$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{B+D}{2}\right)$, and finally,

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{B+D}{2}\right)}$$

Brahmagupta's Theorem: In a cyclic quadrilateral the area
$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $s = \frac{a+b+c+d}{2}$.

Proof. Once Bretschneider's theorem has been proved, Brahmagupta's theorem follows almost immediately, since $\angle B + \angle D = 180$, the last term in Bretschneider's becomes zero.

Problem Set

1. Two circles or radius 4 and 1 are externally tangent. Compute the sine of the angle formed by their common external tangents. **ARML 1986, Team 1**

2. The sides of a quadrilateral are 3, 3, 4 and 8 (in some order). Two of its angles have equal sines but unequal cosines, yet the quadrilateral cannot be inscribed in a circle. Compute the area of the quadrilateral. **ARML 1986, Team 3**

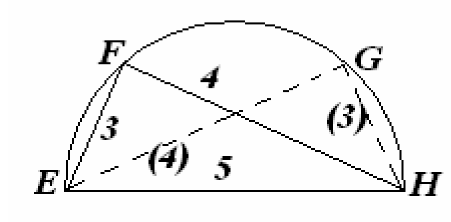
3. Show that if a quadrilateral is cyclic, [that is, it is inscribable in a circle], and its consecutive sides are a,b,c, and d, and its diagonals are p and q, then $pq \le \sqrt{\left(a^2+b^2\right)\left(c^2+d^2\right)}$. **ARML 1987, Power I(c)**

4. A convex n-gon will be called "Pythagorean" if it has integer sides, it is cyclic, and its longest side is a diameter for its circumscribing circle. It shall be denoted by Pn, or Pn:(a,b,…), where a, b, … are the lengths of its sides. We shall always use the letter d for its longest side. [Thus P3 is a Pythagorean triangle. Note that it would be a right triangle.]
   I. [There is a theorem which states (in part) that: If a prime d is the hypotenuse of a Pythagorean triangle, then d^2 is the hypotenuse of two Pythagorean triangles, d^3 is the hypotenuse of three Pythagorean triangles, etc.

   I. A. Find two P3's for which d = 25.
   B. Find three P3's for which d = 125.

   II. Ptolemy's Theorem says: A convex quadrilateral is cyclic if and only if the product of its diagonals equals the sum of the products of the two pairs of opposite sides.

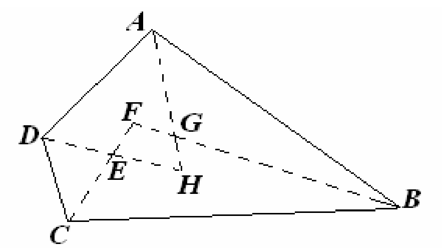
A. If the P3:(3,4,5) is reflected as shown,
A quadrilateral EFGH can be formed (it will not be a P4, as FG is not an integer). Multiplying each side by 5 produces a P4. Find the sides of this P4.



B. Find a P4 with two equals sides and with d = 25 that is different from the answer to part IIA. [Note: Two Pn's are not considered different if their sides are equal, but in a different order.]

C. Show that a Pn must exist for all integers $n \geq 3$. [This may be done by describing how to create such a Pn.]
ARML 1989 Power Question

5. The bisectors of the angles of quadrilateral ABCD are drawn. The form quadrilateral EFGH, as shown, in which $\measuredangle E + \measuredangle F = 193^o$. If $\measuredangle A > \measuredangle C$, compute the numerical value of $\measuredangle A - \measuredangle C$. **ARML 1989 Individual 2**



6. A convex hexagon is inscribed in a circle. If its successive sides are 2, 2, 7, 7, 11, and 11, compute the diameter of the circumscribed circle.
ARML 1989 Individual 8.

7. "Nice" Angles and Polygons:
Definition 1: We call an angle A "nice" if both sin(A) and cos(A) are rational.
Definition 2: We call a convex polygon "nice" if all of its interior angles are "nice."
III. A convex quadrilateral has the following properties:
1. Its sides are integers whose product is a square.
2. It can be inscribed in a circle, and can be circumscribed about (another) circle.
Prove that this quadrilateral is "nice." **ARML 1991 Power Question**