

Algebra II  
State Mathematics Finals  
Solutions

1. B.  $(2x-3)^2 - m = 0 \Rightarrow 4x^2 - 12x + 9 - m = 0 \Rightarrow x^2 - 3x + \frac{9-m}{4} = 0$ , so the sum of the roots is  $-(-3) = 3$ .
2. C.  $\sqrt[x]{\sqrt{3}\sqrt{25}} = 25^{\frac{1}{6x}} = (5^2)^{\frac{1}{6x}} = 5^{\frac{1}{3x}}$
3. B.  $\frac{1}{x+1} > \frac{1}{x-2} \Rightarrow \frac{x-2}{(x+1)(x-2)} - \frac{x+1}{(x+1)(x-2)} > 0 \Rightarrow$   
 $\frac{-3}{(x+1)(x-2)} > 0 \Rightarrow -1 < x < 2$
4. A.  $9^{x+1} + 9^{x+2} + 9^{x+3} + 9^{x+4} + 9^{x+5} = 22143 \Rightarrow 9^x (9 + 9^2 + 9^3 + 9^4 + 9^5) = 22143$   
 $\Rightarrow (3^2)^x (66429) = 22143 \Rightarrow 3^{2x} = \frac{1}{3} \Rightarrow 3^{2x} = 3^{-1} \Rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2}$
5. D.  $\sqrt{1+x+\sqrt{x}} = \sqrt{x+\sqrt{x+7}} \Rightarrow 1+x+\sqrt{x} = x+\sqrt{x+7} \Rightarrow$   
 $1+\sqrt{x} = \sqrt{x+7} \Rightarrow (1+\sqrt{x})^2 = (\sqrt{x+7})^2 \Rightarrow$   
 $1+2\sqrt{x}+x = x+7 \Rightarrow 2\sqrt{x} = 6 \Rightarrow \sqrt{x} = 3 \Rightarrow x = 9$
6. B. Let  $x$  be the number of blue marbles.  
 $\frac{x}{x+80+24} = \frac{1}{5} \Rightarrow 5x = x+104 \Rightarrow x = 26$ .
7. E. Since  $f(x) = f(x-2) + x$ ,  $f(7) = f(5) + 7 \Rightarrow 11 = f(5) + 7 \Rightarrow f(5) = 4$ .
8. A.  $x^4 - 6x^2 + 5 = (x^2 - 5)(x^2 - 1) = (x^2 - 5)(x+1)(x-1)$ , so the sum of the factors is  $(x^2 - 5) + (x+1) + (x-1) = x^2 + 2x - 5$ .
9. D.  $\frac{2x-11}{x^2-5x-14} = \frac{B}{x-7} + \frac{C}{x+2} \Rightarrow$   
 $\frac{2x-11}{(x-7)(x+2)} = \frac{B(x+2)}{(x-7)(x+2)} + \frac{C(x-7)}{(x-7)(x+2)} \Rightarrow$   
 $2x-11 = B(x+2) + C(x-7) \Rightarrow$   
 $2x-11 = (B+C)x + (2B-7C) \Rightarrow B+C = 2$

10. C. To find the inverse, exchange  $x$  and  $y$  in the equation  $y = \frac{x+2}{x-3}$  and solve for  $y$ . Solving for  $y$ ,

$$xy - 3x = y + 2 \Rightarrow xy - y = 3x + 2 \Rightarrow y(x-1) = 3x + 2 \Rightarrow$$

$$y = \frac{3x+2}{x-1}, x \neq 1 \text{ or } f^{-1}(x) = \frac{3x+2}{x-1}, x \neq 1.$$

11. B. 
$$\frac{x^3 - y^3}{x^4 + x^2y^2 + y^4} = \frac{(x-y)(x^2 + xy + y^2)}{x^4 + x^2y^2 + y^4} =$$

$$\frac{(x-y)(x^2 + xy + y^2)}{(x^2 + xy + y^2)(x^2 - xy + y^2)} = \frac{(x-y)}{(x^2 - xy + y^2)}$$

Using long division one can show that  $x^2 + xy + y^2$  is a factor of  $x^4 + x^2y^2 + y^4$ .

12. E. This function has a vertical asymptote at  $x = 0$ , but of more importance to this question is the horizontal asymptote at  $y = 1$ , since  $F(x) = \frac{x+3}{x} = 1 + \frac{3}{x}$ .

Since  $y$  can never equal 1, but can equal anything else, the range of this function is  $\{x \mid x \in \text{Reals}, x \neq 1\}$ .

13. C. 
$$\frac{9+3^{2x}}{10} = 3^x \Rightarrow 3^2 + 3^{2x} = 10 \cdot 3^x \Rightarrow 3^{2x} - 10 \cdot 3^x + 9 = 0 \Rightarrow$$

$$(3^x - 9)(3^x - 1) = 0 \Rightarrow 3^x - 9 = 0 \text{ or } 3^x - 1 = 0 \Rightarrow$$

$$3^x = 9 \text{ or } 3^x = 1 \Rightarrow x = 2 \text{ or } x = 1$$

If  $x = 2$ ,  $x^2 + x + 1 = 7$  and if  $x = 0$ ,  $x^2 + x + 1 = 0$ , so there are 2 solutions, 1 or 7.

14. E. If  $y^2 = |5 - 4y| \Rightarrow y^2 = 5 - 4y$  or  $y^2 = -(5 - 4y) \Rightarrow$

$$y^2 + 4y - 5 = 0 \text{ or } y^2 - 4y + 5 = 0.$$

The first of these,  $y^2 + 4y - 5 = 0 \Rightarrow (y+5)(y-1) = 0$  has solutions -5 and 1 with a sum of 4. The second equation has no real solutions.

15. D.  $x > p$  and  $y > q \Rightarrow (x+y) > (p+y) > (p+q)$ . Let  $x = y = 0$  and  $p = q = -1$ . Then I and iii are false. So ii only is true.

16. E.  $({}_{10}C_2)({}_8C_5)({}_3C_3) = 45 \cdot 56 \cdot 1 = 2520$ .

17. A. For stacks of 8 with 7 bills left over,  $x = 8n + 7$ , for some integer  $n, n \geq 1$ , so  $x \in \{15, 23, 31, 39, \dots, 95\}$ . Similarly,  $x = 6k + 3$ , for some integer  $k, k \geq 1$ , so  $x \in \{9, 15, 21, \dots, 97\}$ , and  $x = 7m + 4$  for some integer  $m, m \geq 1$ , so  $x \in \{11, 18, 25, \dots, 95\}$ . Since 39 is the only element of all 3 sets, we seek to find the remainder when 39 is divided by 5. That remainder is 4.

18. C.  $y = 2x^2$  and  $y = x^2 + x + 6 \Rightarrow 2x^2 = x^2 + x + 6 \Rightarrow$   
 $x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = 3$  or  $x = -2$   
 $\Rightarrow y = 2 \cdot 3^2 = 18$  or  $y = 2(-2)^2 = 9$

The parabolas intersect at (3,18) and (-2,8). The line through (3,18) and (-2,8) has slope  $m = 2$  and equation  $2x - y + 12 = 0$ .

19. E.  $6x^2 + 11x + k = 0 \Rightarrow x^2 + \frac{11}{6}x + \frac{k}{6} = 0$ , therefore  $-\frac{11}{6}$  is the sum of the roots of the equation.

20. A.  $f(i) = \left( \frac{i^4 - i^3 + i^2 - i + 1}{i} \right)^3 = \left( \frac{1 + i - 1 - i + 1}{i} \right)^3 = \left( \frac{1}{i} \right)^3 = \frac{1}{i^3} \cdot \frac{i}{i} = \frac{i}{1} = i$ .

21. C.  $-\log_2(x - \sqrt{x^2 - 1}) = \log_2(x - \sqrt{x^2 - 1})^{-1} = \log_2\left(\frac{1}{x - \sqrt{x^2 - 1}}\right) =$   
 $\log_2\left(\frac{1}{x - \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}\right) = \log_2\left(\frac{x + \sqrt{x^2 - 1}}{x^2 - (x^2 - 1)}\right) = \log_2(x + \sqrt{x^2 - 1})$

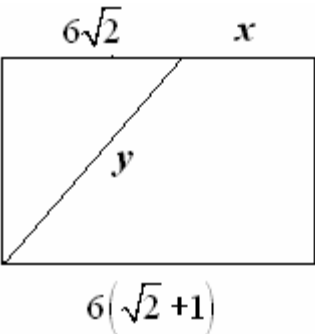
22. A.  $\ln \sqrt[a]{e^5} + \frac{2 \ln \sqrt[b]{e^4}}{a} - \frac{5 \ln \sqrt[a]{e^3}}{3} = 16 \Rightarrow \ln e^{\frac{5}{a}} + \frac{2}{a} \ln e^{\frac{4}{b}} - \frac{5}{3} \ln e^{\frac{3}{a}} = 16$   
 $\Rightarrow \frac{5}{a} + \frac{2}{a} \cdot \frac{4}{b} - \frac{5}{3} \cdot \frac{3}{a} = 16 \Rightarrow 5b + 9 - 5b = 16ab \Rightarrow ab = \frac{1}{2}$

23. B.  $8 \cdot {}_n P_3 = 2 \cdot {}_n P_4 \Rightarrow 4 \cdot {}_n P_3 = {}_n P_4 \Rightarrow 4n(n-1)(n-2) = n(n-1)(n-2)(n-3)$   
 $\Rightarrow 4 = n - 3 \Rightarrow n = 7$

24. C. If  $3w = \frac{x}{2}$ , then  $6w = x$ , so  $F(3w) = (6w)^2 + (6w) + 3 = 9 \Rightarrow 36w^2 + 6w - 6 = 0$   
 $\Rightarrow 6w^2 + w - 1 = 0 \Rightarrow (3w - 1)(2w + 1) = 0 \Rightarrow w = \frac{1}{3}$  or  $w = -\frac{1}{2}$ .

So the sum of these values is  $-\frac{1}{6}$ .

25. B.  $F(2x+5) = F(5) \Rightarrow x = 0$ . When  $x = 0$ , then  $y = 3F(5) - 5 = 3 \cdot 7 - 5 = 16$ .  
 $x + y = 0 + 16 = 16$ .

26. D. 

$$x = 6(\sqrt{2} + 1) - 6\sqrt{2} = 6$$

$$y = 6\sqrt{2} \cdot \sqrt{2} = 12$$

$$\text{Perimeter} = 6 + 6\sqrt{2} + 6(\sqrt{2} + 1) + 12 = 24 + 12\sqrt{2}$$

27. D.  $\log_5 \sqrt{2} = \frac{\log \sqrt{2}}{\log 5} = x \Rightarrow \frac{\log 5}{\log \sqrt{2}} = \log_{\sqrt{2}} 5 = \frac{1}{x}$

$$x^2 - (1 - 2i)x = (\frac{1}{2} + i) \Rightarrow 2x^2 + (-2 + 4i)x - 1 - 2i = 0$$

28. E.  $x = \frac{-(-2 + 4i) \pm \sqrt{(-2 + 4i)^2 - 4(2)(-1 - 2i)}}{4} = \frac{2 - 4i \pm \sqrt{-4}}{4}$   
 $x = \frac{2 - 4i \pm 2i}{4}, x = \frac{1 - i}{2}$  or  $x = \frac{1 - 3i}{2}$

29. C.  $x^{\frac{1}{2}} \cdot x^{-\frac{1}{4}} \cdot x^{\frac{1}{8}} \cdot x^{-\frac{1}{16}} \dots = x^{\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots}$  The exponent in this last expression is an infinite geometric series with sum  $\frac{\frac{1}{2}}{1 - (-\frac{1}{2})} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$ , so

$$x^{\frac{1}{2}} \cdot x^{-\frac{1}{4}} \cdot x^{\frac{1}{8}} \cdot x^{-\frac{1}{16}} \dots = x^{\frac{1}{3}} = \sqrt[3]{x}.$$

30. E. Using substitution with the first two equations and then the first and third equations results in the system of two equations in two variables

$$\begin{cases} kx + (-x + 5) = 17 \\ x + k(-x + 5) = -12 \end{cases} \Rightarrow \begin{cases} kx - x = 12 \\ -kx + x = -2 - 5k \end{cases} \Rightarrow 0 = 10 - 5k \Rightarrow k = 2 \Rightarrow 2k + 1 = 5$$

31. C. To find  $f^{-1}$ ,  $x = y + \frac{1}{4} \Rightarrow 4x = 4y + 1 \Rightarrow y = \frac{4x - 1}{4} \Rightarrow f^{-1}(x) = x - \frac{1}{4}$ . Then

$$(h \circ f^{-1})\left(\frac{1}{2}\right) = h\left(f^{-1}\left(\frac{1}{2}\right)\right) = h\left(\frac{1}{2} - \frac{1}{4}\right) = h\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^{\frac{3}{2}} = \frac{1}{8}.$$

32. A. The 110<sup>th</sup> group has 110 numbers beginning with 5996. So the sum  $S$  is  

$$S = \frac{110}{2}(5996 + 6105) = 665,555.$$

33. C.  $f(a) = b \Rightarrow a^2 = b$  and  $f(c) = d \Rightarrow c^2 = d$ . Also slope  

$$m = \frac{d-b}{c-a} = \frac{c^2-a^2}{c-a} = 5.$$
 Since the x-coordinates of P and Q differ by 1,  
 $c-a \neq 0$ , so  $m = c+a = 5$ . Also  $a+1 = c$ , or  $c+1 = a$ . If  $a+1 = c$ , then  
 $c+a = 5 \Rightarrow 2a+1 = 5 \Rightarrow a = 2$  and  $c = 3$ . If  $c+1 = a$ , then  
 $c+a = 5 \Rightarrow c = 2$  and  $a = 3$ . Then  $b+d = a^2 + c^2 = 2^2 + 3^2 = 4 + 9 = 13$

34. D.  $x^2 + y^2 - 10x + 6y + 9 = 0 \Rightarrow (x-5)^2 + (y+3)^2 = 25$ . So L passes through  
the center (5,-3) and (7,1) and has slope  $m = \frac{-3-1}{5-7} = 2$  and the equation of L is  
 $y = 2x - 13$ . The only choice that lies on the line is (8,3), choice D.

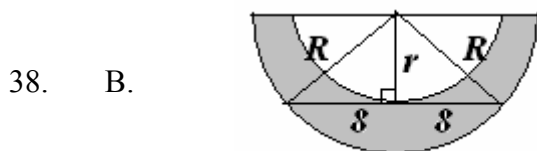
35. A.  $x^4 + 2x^3 + 9x^2 + 18x = 0 \Rightarrow x(x+2)(x^2+9) = 0 \Rightarrow x = 0, -2, \text{ or } \pm 3i$ . The  
sum of the squares of all solutions is  $0^2 + (-2)^2 + (3i)^2 + (-3i)^2 = 4 - 9 - 9 = -14$ .

36. D.  $ax + by = (a-b)^2$  and  $ax - by = a^2 - b^2 \Rightarrow 2ax = a^2 - 2ab + a^2$   
 $\Rightarrow ax = a^2 - ab \Rightarrow x = a - b$  and  $y = b - a \Rightarrow x - y = a - b - (b - a)$   
 $= 2(a - b)$

37. D. By substitution,  

$$7 \cos x + 3 \sin x = 7 \sin x + 3 \cos x \Rightarrow \sin x = \cos x \Rightarrow x = \frac{\pi}{4}$$
  

$$\Rightarrow y = 7 \cos \frac{\pi}{4} + 3 \cos \frac{\pi}{4} = 7 \cdot \frac{\sqrt{2}}{2} + 3 \cdot \frac{\sqrt{2}}{2} = 5\sqrt{2}$$



Let  $r$  = radius of smaller circle and  $R$  = radius of the larger circle. Then  
 $r^2 + 8^2 = R^2 \Rightarrow R^2 - r^2 = 8^2$ . The area of the shaded region = Area of larger  
semicircle - area of smaller semicircle =  $\frac{\pi}{2}R^2 - \frac{\pi}{2}r^2 = \frac{\pi}{2}(R^2 - r^2) = \frac{\pi}{2} \cdot 64 = 32\pi$

39. E.  $\frac{(x^2 - x - 6)}{x - 5} \geq 0 \Rightarrow \frac{(x-3)(x+2)}{(x-5)} \geq 0 \Rightarrow -2 \leq x \leq 3 \text{ or } x > 5$ .

40. D  $a^2 - b^2 = c^2 - d^2 = 9^2 \Rightarrow (a-b)(a+b) = (c-d)(c+d) = 81$ . If  $a, b, c, d$  are distinct positive integers, then  $1 \cdot 81 = 81$  or  $3 \cdot 27 = 81$ . Let  $c-d=1$  and  $c+d=81$ . Then  $c=41$  and  $d=40$ . Let  $a-b=3$  and  $a+b=27$ . Then  $a=15$  and  $b=12$ . So  $a+b+c+d=108$ .