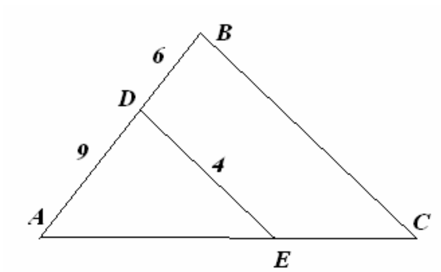
Geometry
State Mathematics Finals Contest Solutions
May 1, 2003

1. B. Since the conditional is true, the contrapositive is also true. The converse, "If a pentagon is equiangular, then it is regular" is false as in the inverse.

2. C. $m\angle AOB = m\angle BOC \Rightarrow 2x+10 = 8x-14 \Rightarrow 24 = 6x \Rightarrow x = 4$.
$m\angle AOC = 2\cdot m\angle AOB = 2(2\cdot 4+10) = 36^\circ$

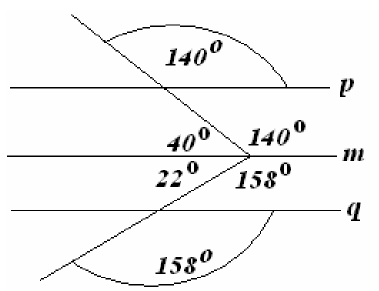
3. C.



$\triangle ADE \sim \triangle ABC$,

$\frac{4}{x} = \frac{9}{15} \Rightarrow x = 6\frac{2}{3}$

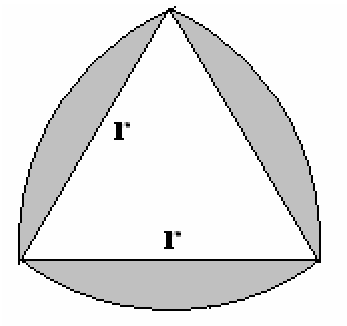
4. D.



Construct a line m so that $m \parallel p$ and $m \parallel q$. Then $x = 40^\circ + 22^\circ = 62^\circ$

5. E. Since $A = \frac{1}{2}bh = 60 \Rightarrow 5h = 60 \Rightarrow h = 12$. Since the altitude of an isosceles triangle bisects the base of the triangle, the legs of the isosceles triangle can be determined by using the Pythagorean Theorem, $5^2 + 12^2 = c^2 \Rightarrow c = 13$. So the perimeter is 2(13) + 10 = 36cm.

6. C.



Consider the triangle and one shaded region. A side of the triangle is a radius of the circle.

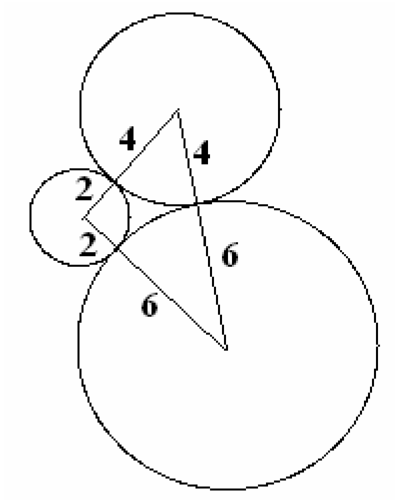
$$Area_{\text{shaded region}} = Area_{\text{sector}} - Area_{\text{triangle}} =$$

$$\frac{60^\circ}{360^\circ}\pi r^2 - \frac{\sqrt{3}}{4}r^2 = \frac{1}{6}\pi r^2 - \frac{\sqrt{3}}{4}r^2 =$$

$$\frac{1}{3}\left(18\pi - 27\sqrt{3}\right) = 6\pi - 9\sqrt{3} \Rightarrow r^2 = 36$$

$$\Rightarrow r = 6$$

7. D.



The sides of the triangle have lengths 2 + 4 = 6, 2 + 6 = 8, and 4 + 6 = 10. Since a 6,8,10 triangle is a right triangle,

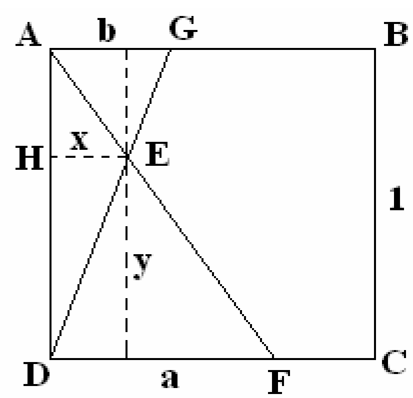
$$A = \frac{1}{2}(6 \cdot 8) = 24$$

8. A. Let x be the desired distance and y be the distance from E to $\overline{DF}$ as shown. Since $\triangle DEF \sim \triangle GEA$,

$$\frac{y}{1-y} = \frac{a}{b} \Rightarrow y = \frac{a}{a+b}.$$

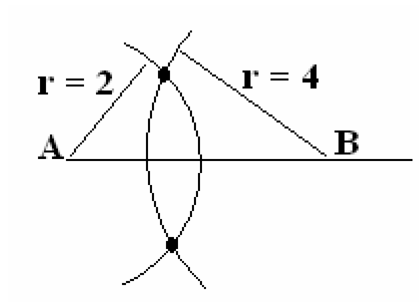
Since $\overline{HE}$ is parallel to side $\overline{AG}$ of $\triangle ADG$, $\frac{x}{b} = \frac{y}{1} \Rightarrow x = by$

$$\Rightarrow x = b\left(\frac{a}{a+b}\right) = \frac{ab}{a+b}.$$



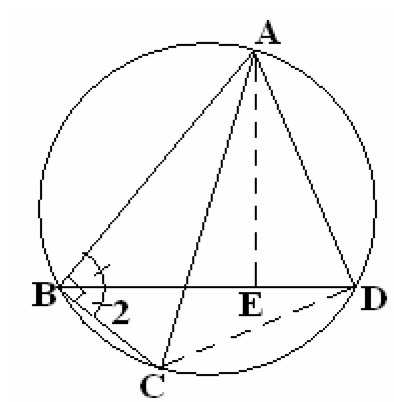
9. B. Since $PQ = 1$ and the radius r of each circle is 2, then $OP = QR = 1$ and $OR = 3$. Then $KL = 2+3+2 = 7$. Since $r = 2$, $LM = 2 \cdot 2 = 4$. The perimeter of $KLMN$ is $P = 2 \cdot 7 + 2 \cdot 4 = 22$.

10. B. A circle with radius 2 and center A will intersect a circle with radius 4 and center B in at most 2 places (if the centers are less than 6 units apart.



11. C. Let A be the measure of one of the angles of the parallelogram and let d be the length of the diagonal opposite this angle. Using the Law of Cosines $d^2 = 5^2 + 12^2 - 2(5)(12)\cos A$. Similarly, if c is the length of the other diagonal, then $c^2 = 5^2 + 12^2 - 2(5)(12)\cos(180 - A) = 5^2 + 12^2 + 2(5)(12)\cos A$. So the sum of these two diagonals is $c^2 + d^2 = \left(5^2 + 12^2 + 2(5)(12)\cos A\right) + \left(5^2 + 12^2 - 2(5)(12)\cos A\right)$, or $c^2 + d^2 = (25+144) + (25+144) = 169 + 169 = 338$.

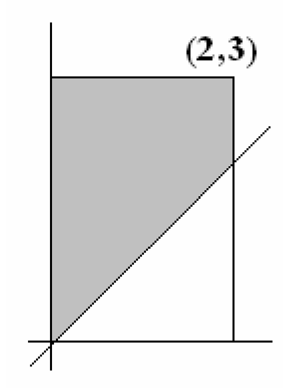
12. B. Draw segment $\overline{CD}$. Since $\overline{AC}$ is a diameter, $\angle ABC$ and $\angle ADC$ are right angles. Because $\angle ABD \cong \angle CBD$ and $\angle ABC$ is a right angle, then $m\angle ABC = m\angle CBD = 45^\circ$. Since $m\angle CBD = 45^\circ$ and $\angle CBD$ and $\angle CAD$ are inscribed angles intercepting $\overset{\frown}{CD}$, $m\angle CAD = 45^\circ$. $\triangle ADC$ is an isosceles right triangle and $AD = CD = \dfrac{AC}{\sqrt{2}}$. Since $\triangle ABC$ is a right triangle, $AC^2 = AB^2 + BC^2 = 20$, so $AC = \sqrt{20}$. Then $AD = CD = \sqrt{10}$. Draw a perpendicular from A to $\overline{BD}$ and let the foot of the perpendicular be E. Then $\triangle ADE$ is a right triangle so $DE^2 = AD^2 - AE^2 = 10 - 8 = 2$, so $DE = \sqrt{2}$. Then $DB = BE + ED = 2\sqrt{2} + \sqrt{2} = 3\sqrt{3}$.



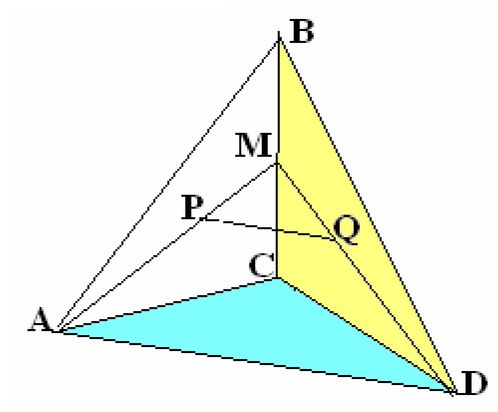
13. A. The reflection of (a,b) about the y-axis is the point (-a,b) and the reflection of that point about the x-axis is the point(-a,-b). So $e = -a, f = -b,$ and $ab - ef = ab - ab = 0$

14. E. Consider the area of the triangle. $A=\frac{1}{2}ab$ and $A=\frac{1}{2}ch$, so $\frac{1}{2}ab=\frac{1}{2}ch \Rightarrow h=\frac{ab}{c}$.

15. C. Draw a rectangle with the given points as vertices and the line with equation $y=x$. All points above the line have x-coordinate less than the y-coordinate. The area of the rectangle is 6 square units, and y = x intersects the right side of the rectangle at the point (2,2). The region above the line y = x is a trapezoid with area $A=\frac{1}{2}(2)(1+3)=4$ square units. The probability that a randomly chosen point is above the line is 4/6 = 2/3.

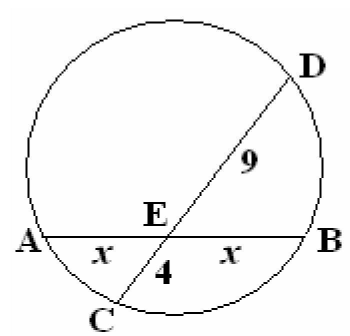


16. C. In $\triangle ABC$, AM one of the medians with length $\frac{\sqrt{3}}{2}$. Similarly, the median DM in $\triangle BCD$ has length $\frac{\sqrt{3}}{2}$. The centroids for each of these triangles is located 2/3rds the way from the vertex to the opposite side, so $PM=QM=\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{6}$. Since

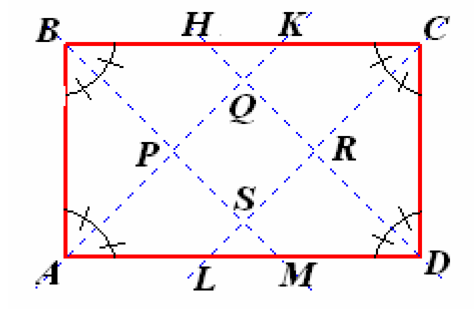


$$\triangle PMC \sim \triangle AMD,\ \frac{PM}{PQ}=\frac{AM}{AD}\Rightarrow\frac{\sqrt{3}/6}{PQ}=\frac{\sqrt{3}/2}{1}\Rightarrow PQ=\frac{1}{3}.$$

17. D. Let $x=AE=EB$. Then $x\cdot x=4\cdot 9\Rightarrow x^2=36\Rightarrow x=6$. Then $AB=2(6)=12$.

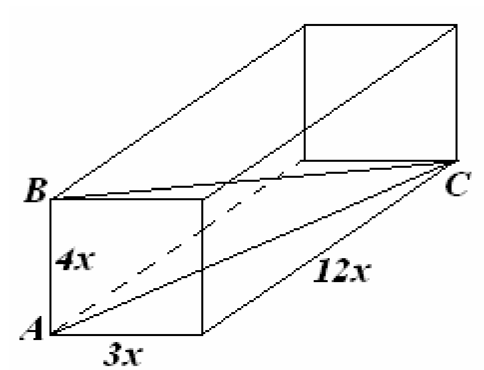


18. A. Draw in the bisectors of the angles and call the points of intersection P, Q, R, S. Since $AK \parallel CL$ and $BM \parallel HD$ the quadrilateral PQRS is a parallelogram. Since $m\angle ABP = m\angle BAP = 45^o$, $m\angle BPA = mQPS = 90^o$, so PQRS is a rectangle. Finally, $\triangle ABK \cong \triangle CDL \cong \triangle MAB \cong \triangle HCD$ and $\triangle HQK \cong \triangle MSL$, $QR = RS = SP = PQ$, so PQRS is a square.



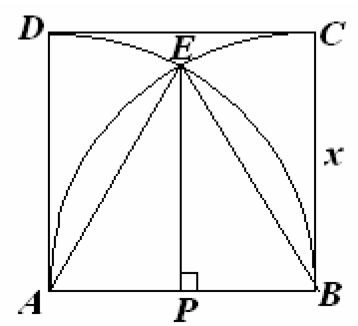
19. D. Let 3x, 4x, and 5x be the measures of the 3 remaining exterior angles. Since the sum of the exterior angles of a polygon is 360^o, $75+105+3x+4x+5x=360$, $180+12x=360 \Rightarrow x=15$, so the smallest angle has measure $3\cdot 15 = 45^o$.

20. C. Let A, B, C be the vertices of the rectangular solid as shown. Then $AC^2 = (3x)^2 + (12x)^2 \Rightarrow AC = \sqrt{153}x$. Also $\left(\sqrt{153}x\right)^2 + (4x)^2 = BC^2 \Rightarrow BC = 13x$. Solving $39 = 13x, x = 3$, so the longest dimension of the solid has length $12x = 12(3) = 36$.

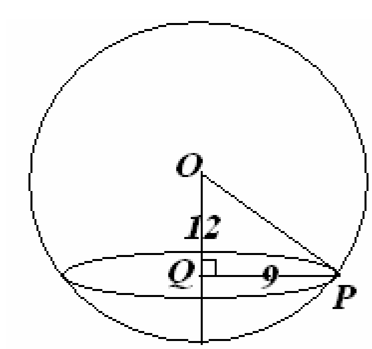


21. D. Draw triangle ABE which is equilateral and construct $\overline{EP}$ so that $\overline{EP} \perp \overline{AB}$. The desired distance from E to CD equals $x - EP$. $EP = \frac{\sqrt{3}}{2}x$ and

$$x - EP = x - \frac{\sqrt{3}}{2}x = x\left(1 - \frac{\sqrt{3}}{2}\right) = x\left(\frac{2-\sqrt{3}}{2}\right) = \frac{x}{2}\left(2-\sqrt{\ }\right)$$



22. D. A right triangle with legs of lengths 9 and 12 and hypotenuse $\overline{OP}$, where $\overline{OP}$ is the radius of the sphere, is formed. So $9^2 + 12^2 = OP^2 \Rightarrow OP = 15$

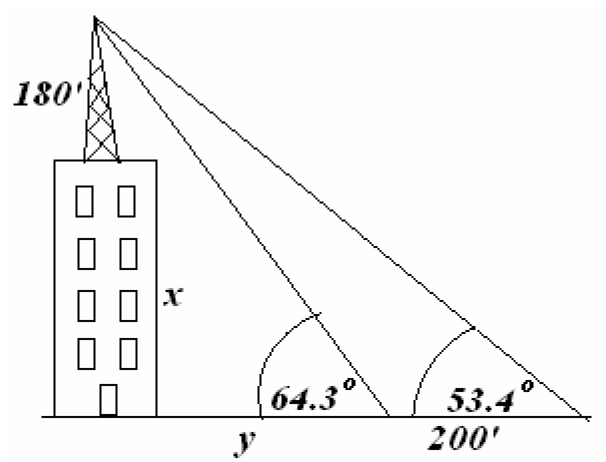


23. D. To determine the difference, subtract the area of the square from the area of the circle. Since a side of the square has length 36", the diameter of the circle is $36\sqrt{2}"$, making the radius $18\sqrt{2}"$. So the desired difference equals $s^2 - \pi r^2 = 36^2 - \pi\left(18\sqrt{20}\right)^2 = 740 \text{ in}^2$.

24. D. The slope m of the radius is $m = \frac{2-(-2)}{7-4} = \frac{4}{3}$, so the slope of the desired line is $-\frac{3}{4}$. The equation of the tangent line through (7,2) is $y - 2 = -\frac{3}{4}(x-7)$ or $4(y-2) = -3(x-7) \Leftrightarrow 3x + 4y = 29$.

25. B. Let the radius of the inner circle be 1 unit. Then the entire dartboard has radius 3 units and the area of the dartboard is $A = \pi(3)^2 = 9\pi$. The area of the region marked 1 or 2 is $A = \pi(4)^2 = 16\pi$, so the probability of hitting the area marked 3 is the ratio of these areas, or $\frac{9\pi - 4\pi}{9\pi} = \frac{5\pi}{9\pi} = \frac{5}{9}$.

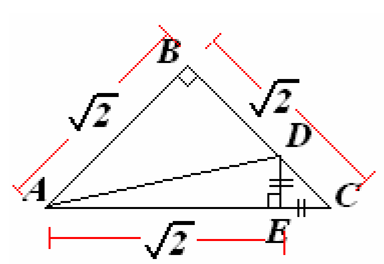
26. E. Let x be the height of the building and y be the distance to the building after moving closer. Then $\tan(64.3^o) = \frac{180+x}{y}$ and $\tan\left(53.4^o\right) = \frac{x+180}{y+200}$. Solving this system of equations for x gives $x \approx 585$ ft.



27. B. $2l + 3x = 132 \Rightarrow l = \frac{132-3x}{2}$.

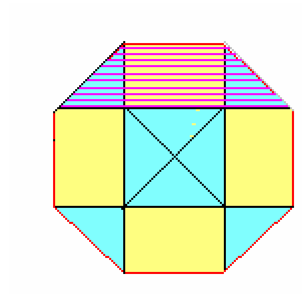
The total area is 572 ft^2, so $572 = x\left(\frac{132-3x}{2}\right) \Rightarrow x^2 - 44x + 384 = 0$, so $x = 12$ or $x = 32$. The difference of these values is 32 – 12 = 20.

28. D. Since the area of $\triangle ABC$ is 1, then $1 = \frac{1}{2}(AB)^2 \Rightarrow (AB)^2 = 2 \Rightarrow AM = \sqrt{2}$. Since $\triangle ABC$ is an isosceles right triangle, $BC = AC = \sqrt{2}$, also $AE = \sqrt{2}$. Then $EC = 2 - \sqrt{2}$ and $DE = 2 - \sqrt{2}$. So the area of $\triangle ADC = \frac{1}{2}(2)(2-\sqrt{2}) = 2 - \sqrt{2}$.



29. C. The surface area of the solid consists of 6 faces of dimension 3 x 3 each with a 1 x 1 square hole. The walls of the 6 holes can each be unfolded to form 1 x 4 rectangles. Thus the surface area is $6(3\cdot 3-1\cdot 1)+6\cdot 4=72$.

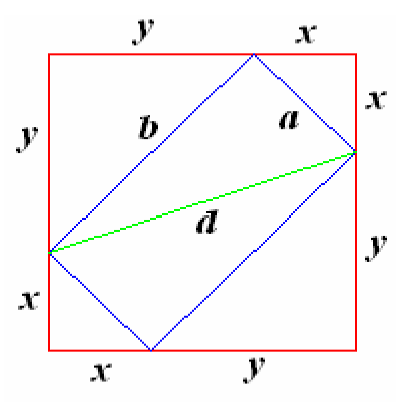
30. B. The diagram shows how the octagon can be decomposed into 4 congruent rectangles and 8 congruent triangles. Let R represent the area of a rectangle and let T represent the are of a triangle. Then the ratio of the shaded region to the area of the entire octagon is $\frac{R+2T}{4R+8T}=\frac{1}{4}$.



31. A. By the Triangle Inequality, $13-9<k<13+9 \Rightarrow 4<k<22$. For an obtuse triangle, $9^2+13^2<k^2 \Rightarrow 250<k^2 \Rightarrow k>15.8$, so $k\in\{16,17,18,19,20,21\}$. Also $9^2+k^2<k^2 \Rightarrow k^2<88 \Rightarrow k<9.38$, so $k\in\{5,6,7,8,9\}$. There are 11 possible values for *k*.

32. B. If Ali broke the toy, then Barbara is lying by saying that Tyler broke it. So Ali cannot have broken the toy. If Tyler broke it, then Ali is lying. So Tyler cannot have broken it. If Hei-Lam broke it, then both Ali and Barbara are lying. So Hie-Lam cannot have broken it, then nobody else is lying. Thus it must have been Barbara who broke it.

33. D. Let x and y be the lengths of the sides of the two isosceles triangles removed from the square, and a and b be the sides of the rectangle that remains. Then the area of the triangles removed are $\frac{1}{2}x^2, \frac{1}{2}x^2, \frac{1}{2}y^2$, and $\frac{1}{2}y^2$.



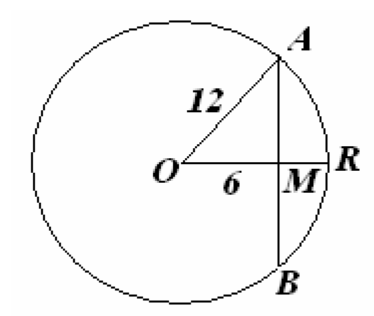
So the total area removed is $x^2+y^2=200$. The lengths of the sides of the rectangle are $a=\sqrt{2}x$ and $b=\sqrt{2}y$. Thus the diagonal of the rectangle is $d^2=a^2+b^2=\left(\sqrt{2}x\right)^2+\left(\sqrt{2}y\right)^2=2\left(x^2+y^2\right)=2(200)=400$. Thus $d=\sqrt{400}=20$

34. D. Let t be the number of minutes since 3:00 when the two hands are perpendicular. This time will be sometime after 3:30 so we expect t to be greater than 30. The minute hand will have moved through an angle of $\frac{t}{60}\cdot 360^o=(6t)^o$. The hour hand will have moved $\frac{t}{60}\cdot 30^o=\left(\frac{t}{2}\right)^o$, so the hour hand is $\left(\frac{t}{2}+90\right)^o$ from 12:00. When the two hands are next perpendicular we have

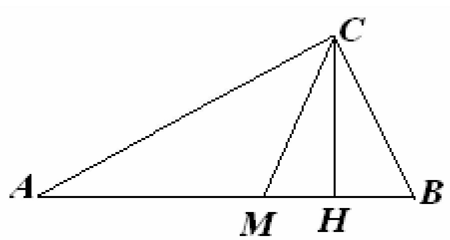
$6t-\left(\frac{t}{2}+90\right)=90 \Rightarrow 6t-\frac{t}{2}=180 \Rightarrow 11t=360 \Rightarrow t=\frac{360}{11}=32\frac{8}{11}$ minutes or approximately 32 minutes and 43.6 seconds.

35. C. Since $\angle DAE+\angle ADE=90^o=\angle DAE+\angle BAE$, we see that $\angle ADE=\angle BAE$ and thus $\triangle ADE$ is similar to $\triangle BAF$. This means that $\frac{DE}{AE}=\frac{AF}{BF}$ or $\frac{5}{3}=\frac{AF}{BF}$. Similarly $\frac{CF}{BF}=\frac{DE}{CE}=\frac{5}{7}$. Therefore $\frac{AF}{BF}+\frac{CF}{BF}=\frac{AF+CF}{BF}=\frac{AE+CE}{BF}=\frac{10}{BF}=\frac{5}{3}+\frac{5}{7}=\frac{50}{21}$, thus $BF=\frac{210}{50}=\frac{21}{5}=4.2$

36. D. Let *O* denote the center of the circle, and let *OR* and *AB* be the radius and the chord which are perpendicular bisectors of each other at *M*. Applying the Pythagorean theorem to the right triangle *OMA* yields $(AM)^2=(OA)^2-(OM)^2=12^2-6^2=108$, Thus $AM=6\sqrt{3}$ and the required chord has length $12\sqrt{3}$.

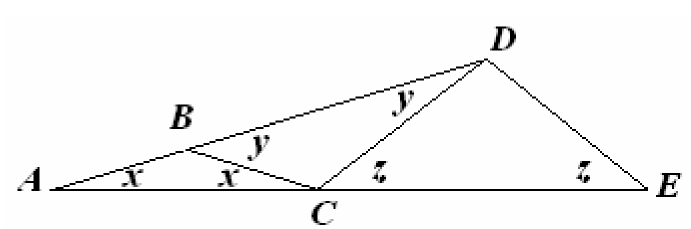


37. E. Right triangles *CHM* and *CHB* are congruent since there angles at *C* are congruent. Therefore the base *MH* of $\triangle CMH$ is $\frac{1}{4}$ of the base AB of $\triangle ABC$, while their altitudes are equal. Hence the area of $\triangle ABC=4K$.



38. E. Let $x=m\angle BAC=m\angle BCA; y=m\angle CBD=m\angle CDB, z=m\angle DCE=m\angle DEC$. Applying the theorem on exterior angles to $\triangle ABC$ and $\triangle ACD$ and the theorem on the sum of the interior angles of a triangle to $\triangle ADE$ yields

$y=2x,\ z+x+y=3x,\ x+m\angle ADE+z=180^o, 140^o+4x=180^o, x=10^o$



39. E. Since $A_1, A_2, A_1 + A_2$ are in arithmetic progression, $A_2 - A_1 = (A_1 + A_2) - A_2 \Rightarrow A_2 = 2A_1$. If r is the radius of the smaller circle, then $9\pi = A_1 + A_2 = 3A_1 = 3\pi r^2 \Rightarrow r = \sqrt{3}$.

40. B. ABC is the given triangle and $x = a$ is the dividing line. Since area of $\triangle ABC = \frac{1}{2}(1)(8) = 4$, the two regions must each have area 2. Since the portion of $\triangle ABC$ to the left of the vertical line through vertex A has area less than area $\triangle ABF = \frac{1}{2}$, the line $x = a$ is indeed right of A as shown. Since the equation of line BC is $y = \frac{x}{9}$, the vertical line $x = a$ intersects BC at a point $E:(a, \frac{a}{9})$. Thus area $\triangle DEC = 2 = \frac{1}{2}\left(1 - \frac{a}{9}\right)(9 - a)$ or $(9-a)^2 = 36$. Then $9 - a = \pm 6$, so $a = 15$ or 3. Since the line $x = a$ must intersect since AC, $x = 3$.