NC STATE MATHEMATICS CONTEST – APRIL 2008

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. If Adam had three more quarters, he would have twice as many quarters as Barbara. If Barbara had six more quarters, she would have three times as many quarters as Adam. If Adam and Barbara put all of their quarters together, how many quarters would they have?

a) 6 b) 8 c) 10 d) 12 e) 16

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2. If $0 < a < b$, how many real solutions, x, does the equation $\pi|x-a| = 2|x-b|$ have?

a) 1 b) 2 c) 3 d) 4 e) The equation has no real solutions.

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3. How many solutions, x, does the equation $\tan(2x) = \cot(x)$ have if $0 \le x \le 2\pi$?

a) 3 b) 4 c) 5 d) 6 e) More than 6

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4. Let a be a fixed real number that is greater than 1. How many real numbers b are there such that the equation $a^x + a^{-x} = b$ has a unique real solution x ?

a) There are no such values of *b*.
b) There is exactly one such value of *b*.
c) There is more than one but finitely many such values of *b*.
d) There are infinitely many such values of *b*.
e) The number of such values of *b* depends on the value of *a*.

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5. For non-zero c, let $f_c(t) = \dfrac{t+1}{ct-1}$ for $t \neq \dfrac{1}{c}$. Let I be the set of non-zero real numbers, c, such that $f_c(f_c(t)) = t$ for all $t \neq \dfrac{1}{c}$. Which of the following statements about the set I is correct?

a) *I* contains no real numbers.
b) *I* contains exactly one real number.
c) *I* contains more than one, but finitely many real numbers.
d) *I* contains infinitely many, but not all, non-zero real numbers.
e) *I* contains all non-zero real numbers.

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6. How many distinct intersections do the two polar curves $r = \dfrac{1}{\cos(\theta)}$ and $r = \dfrac{1}{\cos(\theta - \pi/4)}$ have in the xy-plane?

a) 0 b) exactly 1 c) exactly 2 d) exactly 3 e) more than 3

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7. Two coins are removed at random and without replacement from a box that contains 3 nickels, 2 dimes and 1 coin whose value is 0. What is the probability that the value of one of the two removed coins is five cents more than the value of the other removed coin?

a) 1/3 b) 2/5 c) 7/15 d) 1/2 e) 3/5

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8. In a two-person chess tournament, the contestants play a total of 21 games. The winner of a game is awarded 1 point and the loser is awarded 0 points. If a game ends in a tie, both players are awarded 1/2 point. If Contestant I won 4 more games than her opponent, how many points, altogether, were awarded to Contestant I?

a) 25/2 b) 12 c) 23/2 d) 11

e) The answer cannot be determined uniquely from the given information.

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9. A disk, D, is inscribed in a square, S, with edge length 2. The four points of contact between the circle and the square form the vertices of a smaller square T. Two points are chosen at random and independently from S. A point in T earns 8 dollars, a point in D but not in T earns 9 dollars, and a point in S but not in D earns 10 dollars. What is the probability that the two chosen points will earn a total of eighteen dollars?

a) 0.189 b) 0.286 c) 0.296 d) 0.378 e) 0.592

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10. Given the points O(0, 0), A(0, 1), and B(1, 1) in the xy–plane, suppose that points C(x, 1) and D(1, y) are chosen such that $0 < x < 1$ and such that points O, C, and D are collinear. If x has also been chosen such that the sum of the areas of triangles OAC and BCD is as small as possible then

a) $x = 1/2$.

b) x is a rational number greater than 1/2 and less than or equal to 2/3.

c) x is an irrational number between 1/2 and 2/3.

d) x is a rational number greater than 2/3 and less than or equal to 1.

e) x is an irrational number between 2/3 and 1.

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11. A *palindrome* on the alphabet {H,T} is a sequence of H's and T's which reads the same from left to right as from right to left. Thus, HTH, HTTH, HTHTH and HTHHTH are palindromes of lengths 3, 4, 5 and 6 respectively. Let $P(n)$ denote the number of length n palindromes on the alphabet {H,T}. For how many values of n is $1000 < P(n) < 10{,}000$?

a) 4 b) 5 c) 6 d) 7 e) 8

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12. Let $n = 77553311$ <u>in base 16</u>, i.e. $n = 7(16^7) + 7(16^6) + 5(16^5) + \cdots + 3(16^2) + 1(16) + 1(1)$. Let S be the set of <u>distinct</u> base 16 numbers that are obtained by rearranging the digits in n. How many x in S have the property that $n - x$ is divisible by ten? [For example, if $x = 75753311$ in base 16, then $n - x$ is divisible by 10 since its base 10 representation ends in 0.]

a) 630 b) 1260 c) 2520 d) 20,160 e) 40,320

13. The symbols *a, b, c, d, e, f* represent the integers 1, 2, 3, 4, 5, 6 in some order. How many orderings of the integers from 1 through 6 satisfy both of the following equations?

$$a+b+c+d=e+2f \text{ and } 2a+b=2d+c$$

a) 0 b) 1 c) 2 d) 3 e) 4

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14. The entries in the two-by-two determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ are integers that are chosen randomly and independently, and, for each entry, the probability that the entry is odd is p. If the probability that the value of the determinant is even is 1/2, what is the value of p ?

a) 1/3 b) 1/2 c) 2/3 d) $\sqrt{2}/2$ e) None of a) through d) is correct.

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15. Given a unit cube, let T be the set of all triangles whose vertices are also vertices of the cube. How many non-congruent triangles are contained in T ?

a) 2 b) 3 c) 4 d) 5 e) More than 5

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16. How many four-digit integers, n, have the property that the value of $3n$ is the four-digit integer obtained by writing the digits of n in reverse order?

a) 0 b) 1 c) 2 d) 4 e) More than 4

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17. One side of a parallelogram has length 3, and another side has length 4. Let a and b denote the lengths of the diagonals of the parallelogram. Which of the following quantities can be determined from the given information?

$$(I)\ a+b,\ (II)\ a^2+b^2,\ (III)\ a^3+b^3$$

a) Only (I) b) Only (II) c) Only (III)

d) Only (I) and (II) e) None of the three

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18. Alice has a coin that lands heads with probability 1/3 and Bill has a coin that lands heads with probability 1/4. Alice and Bill use these coins to play a game that consists of successive rounds. In each round, the players toss their coins simultaneously. If they both obtain a head, the game is declared to be a tie and the game is over. If exactly one of the two obtains a head, that player is declared to be the winner and the game is over. If neither obtains a head, they play another round. What is the probability that the game ends in a tie?

a) 1/16 b) 1/12 c) 1/6 d) 5/12 e) 7/12

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19. The pair of two–digit numbers 12 and 63 have the interesting property that if the product expression 12×63 is read in reverse order 36×21, the two product expressions are equal, i.e. $12 \times 63 = 36 \times 21$. How many pairs of two-digit numbers with this property are there in which one of the numbers in the pair is greater than 20 and less than 30, and the units digit of the other number in the pair is an odd prime? [NOTE: 1 is not a prime.]

a) 3 b) 4 c) 5 d) 6 e) More than 6

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20. For $n > 4$, let M denote the minimum of $C(a,2)+C(b,2)$ where $a+b=n$ and $2 \le a \le n-2$. [Here $C(x,y)$ denotes the number of ways to choose y objects from x objects.] Consider the following statements:

(I) M is a perfect square whenever n is odd.

(II) M is NOT a perfect square whenever n is even.

(III) No value of M is prime.

Which of (I), (II) and (III) are true?

a) None are true b) Only (I) is true. c) Only (II) is true.

d) Only (I) and (II) are true. e) All three are true.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Suppose that nine men, mowing at equal rates, can mow the grass on a three-acre lot in two hours. How long, <u>in minutes</u> (to the nearest minute), would it take two of these men to mow the grass on a quarter-acre lot?

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2. If the complex numbers x and y satisfy $x^3 - y^3 = 98i$ and $x - y = 7i$, then $xy = a + bi$ where a and b are real numbers. What is the value of $a+b$?

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3. Consider the <u>non-convex</u> quadrilateral $ABCD$ in the xy-plane, where $A = (0,6)$, $B = (10,0)$, $C = (16,20)$ and $D = (8,6)$. What is the area of $ABCD$, expressed to the nearest integer?

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4. If a man rides his bike from home to work at a constant rate of 15 miles per hour, he will arrive at work one hour earlier than if he had walked to work at a constant rate of 3 miles per hour. What is the distance from the man's home to work, <u>expressed in feet</u>, to the nearest foot? [NOTE: There are 5280 feet in a mile.]

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5. How many ordered pairs of integers (i,j) with $1 \le i < j \le 12$ are there such that the greatest common divisor of $2^i - 1$ and $2^j - 1$ is one?

6. A sphere is inscribed in a regular tetrahedron with edge length 2. Thus the sphere is tangent to each of the four faces of the tetrahedron. Let P and Q be two of the points of tangency. If d is the distance between P and Q and $d^2 = m/n$ where m and n are positive integers with greatest common divisor 1, what is the value of $m + n$?

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7. How many positive integers less than or equal to 2008 can be written in the form $n^2 - m^2$ where $1 \le m < n$ are integers?

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8. Six is the smallest positive integer that is the area of a triangle whose edge lengths are consecutive positive integers. If S is the set of all positive integers, n, such that n is the area of a triangle whose edge lengths are consecutive positive integers, what is the smallest integer in S that is greater than six?

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9. At "Genius U." there are 100 male and 100 female physics majors. Of these, 60 men and 40 women have signed up for Professor Faraday's course in quantum mechanics. Professor Faraday also offers a second "invitation only" class in space–time anomalies for physics majors. The number of men and the number of women in the space–time anomalies course must be equal, and the number of students who belong to only one of his two classes must be as small as possible. Assuming that these conditions are satisfied, how many different class sizes are possible for Professor Faraday's space–time anomalies class?

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10. An analog clock is manufactured with an hour hand and a minute hand that are indistinguishable from one another. (There is no second hand on the clock.) At some point in time between noon and midnight, a photograph of the clock face is to be taken. At how many such times will it be impossible to discern the time the photograph was taken from the image of the clock face? (Assume that the position of the clock's hands can be determined with complete accuracy.)

The following problem, which extends Integer Answer Problem #5, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

How many ordered pairs of integers (i, j) with $1 \le i < j \le 27$ are there such that the greatest common divisor of $2^i - 1$ and $2^j - 1$ is one?