State Mathematics Algebra II Contest
May 3, 2001
Solutions

1. (c) $(L+1)(W+1)=84$ and $(L-1)(W-1)=48$ so we get $\begin{array}{l} LW+L+W+1=84 \\ LW-L-W+1=48 \end{array}$.
Subtracting these last two equations yields $2L+2W=36$, so we could go on and solve for L and W, however, we are looking for the perimeter, so we are done.

2. (e) The terms of the sequence, all in terms of w and x are $\cdots, w, x, w+x, w+2x, 2w+3x, 3w+5x, \cdots$. Thus $2w+3x=0$ and $3w+5x=1$. Solving this system of equations gives $w=-3, x=2$, so the terms are $\cdots, -3, 2, -1, 1, 0, 1, \cdots$, making the quantity $2(w+x+y+z)=2(-3+2-1+1)=-2$.

3. (e) $\sqrt{\dfrac{2^{x+2}-2\left(2^{x+1}\right)}{2\left(2^{x+3}\right)}}=\sqrt{\dfrac{2^{x+4}-2^{x+2}}{2^{x+4}}}=\sqrt{\dfrac{2^{x+4}\left(1-2^{-2}\right)}{2^{x+4}}}=\sqrt{1-\frac{1}{4}}=\sqrt{\dfrac{3}{4}}=\dfrac{\sqrt{3}}{2}$.

4. (b) The only case not to be considered is if both are tails, and this occurs with probability of (0.75)(0.40) = 0.30. Thus there will be at least one head 70% of the time.

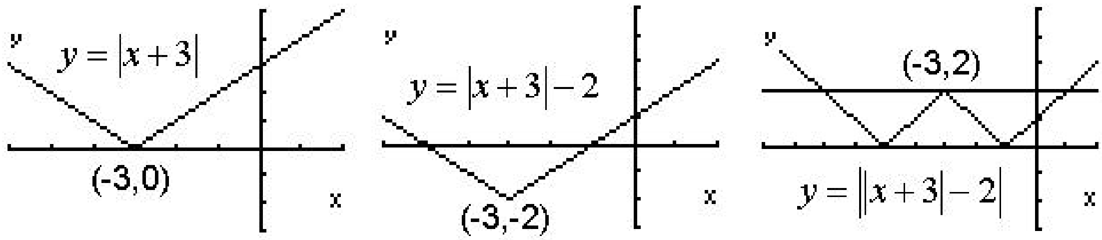
5. (a) If three of the zeros are $2, 1+i, 3-i$, and the coefficients are real, then the conjugates of the imaginary roots must also be zeros. Thus the minimum degree of the polynomial will be five.

6. (d) With the three points (0,-1), (1,4) and (2,13), we could find all of the coefficients of the quadratic, however, since we are looking for the sum of the coefficients, we can plug (1,4) into $y=ax^2+bx+c$ and get $4=a\cdot 1^2+b\cdot 1+c$.

7. (d) $8^{6x^2+4x}=4^{9x^2-9x+6} \Rightarrow 2^{3(6x^2+4x)}=2^{2(9x^2-9x+6)}$. Now with the bases the same we know that the exponents must be the same, so $3(6x^2+4x)=2(9x^2-9x+6)$. This simplifies to $30x=12$ of $x=\frac{2}{5}$.

8. (b) Looking at the amount of grape juice, we get $d\cdot\dfrac{d}{100}+x=(d+x)\dfrac{3d}{100}$, where x is the amount of pure grape juice to be added. This simplifies to $d^2+100x=3d^2+3dx$. Solving for x we get $x=\dfrac{2d^2}{100-3d}$.

9. (c) The following sequence of graphs gives a progression that leads directly to the answer.

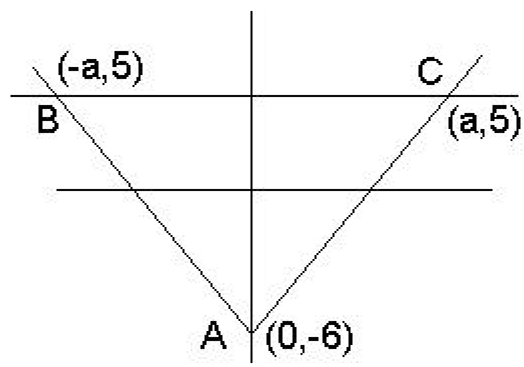


10. (e) $\frac{2}{3}x = (7+9+5+3) \Rightarrow x = 36$

11. (d) $h(x+1) = \frac{3h(x)+4}{3} = h(x) + \frac{4}{3}$, so each term is just four-thirds greater than the one before it. So $h(1) = -\frac{2}{3}, h(2) = -\frac{2}{3} + \frac{4}{3} = \frac{2}{3}$, and $h(3) = \frac{2}{3} + \frac{4}{3} = 2$.

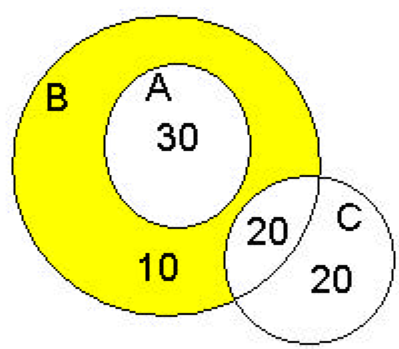
12. (a) Factoring, we get $x^2 + 3xy + 2y^2 = 40 \Leftrightarrow (x+y)(x+2y) = 40$, but since $x + y = 5$, we know that $(x+2y) = 8$ and $2x + 4y = 16$.

13. (b) For the triangle to be equilateral, the sides must all have the same length, so BC = AC, or $2a = \sqrt{a^2 + 11^2}$. Solving for a we get $a = \frac{11}{\sqrt{3}}$, so the slope of side AC is $m = \frac{11}{\frac{11}{\sqrt{3}}} = \sqrt{3}$.



14. (a) $y^* = y^2 - 1 \Rightarrow (y^*)^* = (y^2 - 1)^* = (y^2 - 1)^2 - 1 = y^4 - 2y^2$.

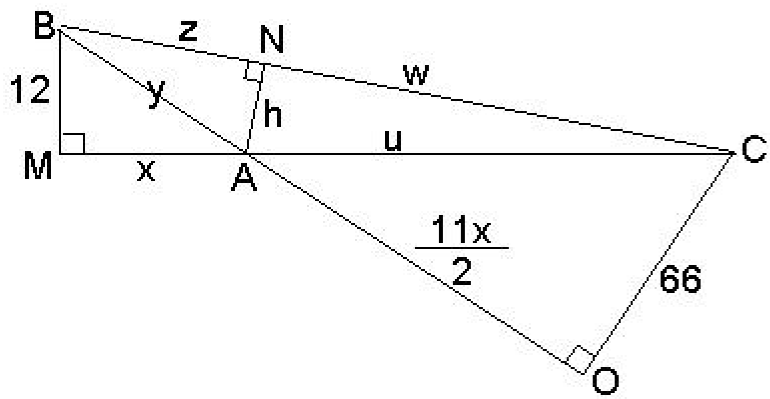
15. (a) The following Venn diagram satisfies the given conditions. The shaded region is outside of A but within B and also outside of C. This is the desired region.



16. (e) $41 + \triangleright 33 \triangleleft - \langle\langle 33 \rangle\rangle + \langle\langle \triangleright 23 \triangleleft \rangle\rangle =$ $41 + 31 - 37 + \langle\langle 19 \rangle\rangle = 35 + 23 = 58$.

17. (d) Let $g(x) = h(x) - 7$, where $h(x) = ax^7 + bx^3 + cx$. Notice that *h(x)* is odd, so $h(5) = -h(-5) = -12$. Thus $g(5) = h(5) - 7 = -12 - 7 = -19$.

18. (d) One extreme for the shortest altitude will occur when the two shortest altitudes are both 12, in an isosceles triangle. The other extreme occurs in a figure similar to the one below. From similar triangles



$\triangle BMA \simeq \triangle COA$ that $\frac{12}{x} = \frac{66}{AO} \Rightarrow AO = \frac{11x}{2}$. Also from similar triangles $\triangle BMC \simeq \triangle ANC$, we know that $\frac{12}{x+u} = \frac{h}{w} \Rightarrow h = \frac{12w}{x+u}$. But we know that $u > w$ and $\frac{11x}{2} < u$, so $h = \frac{12w}{x+u} < \frac{12u}{x+u} < \frac{12u}{\frac{2}{11}u+u}$. Simplifying this we have $h < \frac{12u}{\frac{13u}{11}} = \frac{132u}{13u} = 10\frac{2}{13}$. Thus $12 \le x < 10\frac{2}{3}$, and the only distinct integer length for the shortest altitude is 11.

19. (c) Let M and N be the two numbers. The problem states that M must be a two digit number, but does not indicate the N, the number formed when the digits are reversed, also must be a two digit number. So, let $M = 10a + b$, where $0 < a \le 9$ and $0 \le b \le 9$. Then $N = 10b + a$, and the difference $M - N = (10a+b) - (10b+a) = 9(a-b)$. Now for *M* - *N* to be a perfect cube, *a* - *b* must be a multiple of 3, so we have, for M, 96, 85, 74, 63, 52, 41, and 30.

20. (e) We find that side AC and AB both have length 5, so the angle-bisector is also the median, passing through A(1,-2) and the midpoint (3/2,3/2), of side BC. This makes the slope 7 and the equation $y = 7(x-1) - 2 = 7x - 9$.

21. (b) To find the inverse, exchange the x and the y and solve.
$x = \frac{3y-5}{2y+1} \Rightarrow 2xy + x = 3y - 5 \Rightarrow (2x-3)y = -x-5$, so $y = \frac{x+5}{-2x+3}$, making the product $5 \cdot (-2) \cdot 3 = -30$.

22. (a) Rearrange the terms so that $24{,}130 = 2\cdot 10^b + 4\cdot 10^d + 1\cdot 10^a + 3\cdot 10^c$. Now *b = 4*, *d = 3*, *a = 2*, and *c = 1*, making $\frac{a}{2}+\frac{b}{4}+\frac{c}{8}+\frac{d}{16}=\frac{2}{2}+\frac{4}{4}+\frac{1}{8}+\frac{3}{16}=2\frac{5}{16}$.

23. (e) Setting $11x+3x^2=11x^2-3x \Rightarrow 8x^2-14x=0 \Rightarrow x=0,\frac{7}{4}$. Since we are told that x is greater than zero, we accept only the 7/4.

24. (a) $(a+b)=3 \Rightarrow (a+b)^2=9$, so $a^2+2ab+b^2=9$, but we are also told that $a^2+b^2=6,$ so $2ab=3$. $3^3=(a+b)^3=a^3+3a^2b+3ab^2+b^3$, so $27=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab(a+b)$, and $a^3+b^3=27-3ab(a+b)=27-3\left(\frac{3}{2}\right)(3)=\frac{27}{2}$.

25. (c) $\log_4(8\cdot 2^x)=\log_4 8+\log_4 2^x=\frac{3}{2}+\frac{\log_2 2^x}{\log_2 4}=\frac{3}{2}+\frac{x}{2}$.

26. (b) $x>4-\frac{7}{x+4} \Leftrightarrow x-4+\frac{7}{x+4}>0$. Now simplify the last inequality to $\frac{(x-4)(x+4)+7}{(x+4)}>0$, or $\frac{x^2-9}{(x+4)}>0$. Now factor. $\frac{(x-3)(x+3)}{(x+4)}>0$. From here we see that on the interval [-5,15], the expression on the left is positive on the intervals (-4,-3) and (3,15). This represents 13/20ths of the entire interval, or 65%.

27. (a) $x-y=4\sqrt{3}$ and $xy=8$, so $\left|\frac{1}{x}-\frac{1}{y}\right|=\left|\frac{x-y}{xy}\right|=\frac{4\sqrt{3}}{8}=\frac{\sqrt{3}}{2}$.

28. (e) The discriminant of $ax^2+2bx+c=0$ is $4b^2-4ac$, which factors as $4\left(b^2-ac\right)$. If this is zero, then $b^2=ac$, or $\frac{b}{a}=\frac{c}{b}$, making b the geometric mean between a and c.

29. (a) $\frac{e^x-e^{-x}}{2}=2 \Leftrightarrow e^x-e^{-x}=4$. Now let $w=e^x$ and rewrite the expression as $w+w^{-1}=4 \Leftrightarrow w^2-4w-1=0$, and solve for w. $w=\frac{4\pm\sqrt{20}}{2}=2\pm\sqrt{5}$. Since $w=e^x$, it must be positive, so $e^x=2+\sqrt{5}$ making $x=\ln\left(2+\sqrt{5}\right)$.

30. (a) If you expand the expression $\left(\sqrt[3]{4}+\sqrt[3]{2}-2\right)\left(a\sqrt[3]{4}+b\sqrt[3]{2}+c\right)=20$, you get $2^{\frac{2}{3}}\left(c+b-2a\right)+2^{\frac{1}{3}}\left(c+2a-2b\right)+2\left(a+b-c\right)=20$. Since we know that a, b, and c are integers, the first two terms must be zero, and the final term, $2(a+b-c)$ must equal 20, so $a+b-c=10$. Note: You can solve for a, b, and c, getting $(a,b,c)=(6,8,4)$, but there is no need to since the problem asks for $a+b-c$.

31. (e) If we write the fractions in order as $\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\cdots+\left(\frac{1}{100}+\frac{2}{100}+\cdots+\frac{99}{100}\right)$, we see that this sum is $\left(\frac{1}{2}\right)+\left(\frac{3}{3}\right)+\left(\frac{6}{4}\right)+\left(\frac{10}{5}\right)+\left(\frac{15}{6}\right)+\cdots+\left(\frac{\frac{10099}{2}}{100}\right)$, using the formula for the sum of n-consecutive integers $1+2+3+\cdots+n=\frac{n(n+1)}{2}$. If you now factor one-half from each term you get $\frac{1}{2}(1+2+3+4+\cdots+99)=\frac{1}{2}\left(\frac{99\cdot100}{2}\right)=2475$.

32. (c) Since we know that the area is $a^2+b^2-c^2$, we know that c cannot be the hypotenuse, it must be a or b. We know from the question that a is the hypotenuse. Now we know that $b^2+c^2=a^2$ and that the area is $\frac{1}{2}bc$, so $\frac{1}{2}bc=a^2+b^2-c^2 \Rightarrow \frac{1}{2}bc=2b^2$ since $a^2-c^2=b^2$, Thus $\frac{c}{b}=4$.

33. (a) Let t be time, so we are told that $2\left(1-\frac{1}{5}t\right)=1\left(1-\frac{1}{7}t\right)$. Thus $2-\frac{2}{5}t=1-\frac{1}{7}t \Leftrightarrow 1=\frac{2}{5}t-\frac{1}{7}t=\frac{9}{35}t$. So $t=\frac{35}{9}=3.\bar{8}$ hours.

34. (b) Since the remainder is a constant, P(x) must be quadratic. Look first at the simple quadratic $P(x)=x^2+bx+c$. Since $x+1$ yields the remainder of -5 when divided into P(x), we know that $P(-1)=1-b+c=-5$. We also know that $P(5)=25+5b+c=7$. Solving for b and c we get $b=-2$ and $c=8$. Now the product x^2-4x-5 divides into $P(x)=x^2-2x-8$ one time with a remainder of $2x-3$. Note: If you use a more complicated quadratic for P(x), so $P(x)=ax^2+bx+c$, you find that any a will work, and with $b=2-4a$ and $c=-3-5a$, the product x^2-4x-5 divides into P(x) a times with the same remainder.

35. (a) Complete the squares to find the center of the circle.
$x^2 - 6x + 9 + y^2 - 4y + 4 = -11 + 9 + 4 = 2$, so $(x-3)^2 + (y-2)^2 = 2$, putting the center at (3,2). This point is $\sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13}$ from the origin.

36. (a) $\log_2(a^3 b) = x \Leftrightarrow 3\log_2 a + \log_2 b = x$.
$\log_2\left(\frac{3a}{b}\right) = y \Leftrightarrow \log_2 3 + \log_2 a - \log_2 b = y$. Adding these last two expressions give us $4\log_2 a + \log_2 3 = x + y$. Solving for $\log_2 a$ yields
$\log_2 = \frac{x+y}{4} - \frac{1}{4}\log_2 3 = \frac{x+y}{4} - \log_2 \sqrt[4]{3}$.

37. (c) $\begin{array}{c} 2x + 3y = A \\ x + 2y = B \end{array} \Leftrightarrow \begin{array}{c} 2x + 3y = A \\ 2x + 4y = 2B \end{array} \Rightarrow y = 2B - A$. But
$x = B - 2y \Rightarrow x = B - 2(2B - A) = 2A - 3B$.

38. (d) When you expand $(a+b)^8$ you get a nine terms, each with some coefficient and a and b raised to powers from 0 to 8. Plugging 1 in for both a and b would then give you the sum of the coefficients, to go ahead and plug 1 in at the beginning, getting $(1+1)^8 = 2^8 = 256$.

39. (d) To find the inverse, exchange the x and y and solve for y. Thus
$x = \frac{y+1}{y-1} \Leftrightarrow xy - x = y + 1 \Leftrightarrow xy - y = x + 1 \Leftrightarrow y = \frac{x+1}{x-1}$. From this we see that the function is its own inverse. If we had graphed it, and noticed the symmetry about the line y = x, we could have drawn the same conclusion.
Thus $f^{-1}\left(\frac{1}{x}\right) = f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}+1}{\frac{1}{x}-1} = \frac{x+1}{x-1}$. Setting this equal to 3, we have
$\frac{x+1}{x-1} = 3 \Leftrightarrow x + 1 = 3(x-1) \Rightarrow 2x = 4$, so $x = \frac{1}{2}$.

40. (c) Let the y-intercept by (0,k) and the x-intercept (p,0). Then
$\frac{k-3}{0-4} = \frac{3-0}{4-p} \Rightarrow (k-3)(p-4) = 12$. The positive integer factors of 12 are 1, 2, 3, 4, 6, and 12, so $\begin{array}{c} p-4 = 1,2,3,4,6,12 \\ k-3 = 12,6,4,3,2,1 \end{array} \Rightarrow \begin{array}{c} p = 5,6,7,8,10,16 \\ k = 15,9,7,6,5,4 \end{array}$. Of these, only $p = 5, 7$ are prime.