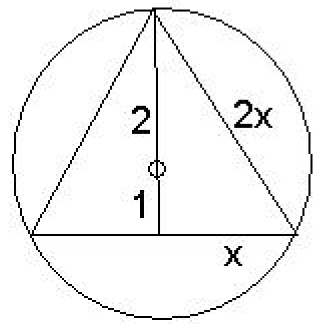
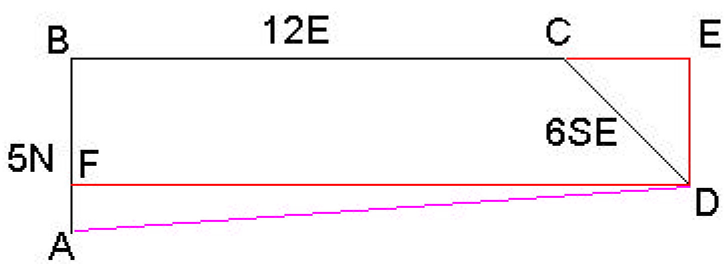
State Mathematics Contest: Geometry
May 3, 2001
Solutions

1. (b) In an equilateral triangle, the centroid, as well as the incenter and circumcenter, is located two-thirds of the distance from the vertex to the opposite side. Then $(2x)^2 = x^2 + 3^2 \Rightarrow 3x^2 = 9 \Rightarrow x = \sqrt{3}$. This makes the area $\frac{1}{2}(2\sqrt{3})3 = 3\sqrt{3}$.



2. (b) Placing the runners on a number line, we see that there is a spread of 20 seconds from first to last, that Jo was at the end, Sam in second 19 second ahead of Jo and one second behind Chris, the winner. Pat then was in third 12 seconds behind Sam.

3. (d) In the figure, we see that $CE = DE = 3\sqrt{2}$. Also $FD = BC + CE = 12 + 3\sqrt{2}$ and $AF = 5 - 3\sqrt{2}$. Thus the distance we want, $AD = \sqrt{AF^2 + FD^2} = \sqrt{\left(12+3\sqrt{2}\right)^2 + \left(5-3\sqrt{2}\right)^2} \approx 16.26$



4. (a) Points on both circles must satisfy both equations, so both (1) $(x-1)^2 + (y+2)^2 = 9$ and (2) $(x+3)^2 + (y-1)^2 = 16$. Squaring both, and subtracting gives $\left(x^2+6x+9+y^2-2y+1\right) - \left(x^2-2x+1+y^2+4y+4\right) = 16-9$, or simply $8x - 6y = 2 \Rightarrow 4x - 3y = 1$

5. (d) The new length and width are 400 + 2w and 300 + 2w, so the new area is $(400+2w)(300+2w) = 2(400)(300) \Rightarrow$

$$120000 + 1400w + 4w^2 = 240000 \Rightarrow$$

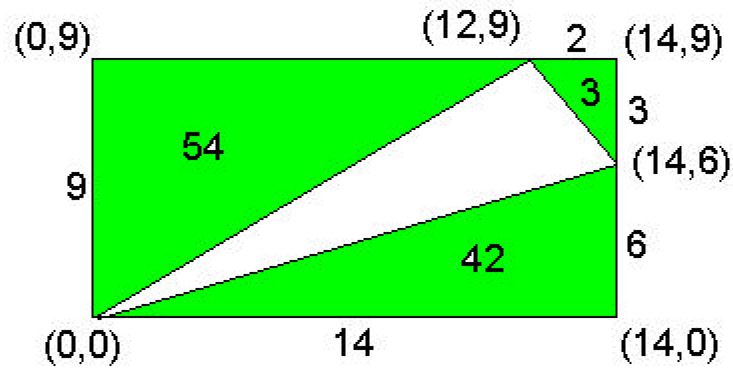
$$w^2 + 350w - 30000 = 0 \Rightarrow$$

$$w = \frac{-350 + \sqrt{(-350)^2 - 4(-30000)}}{2} \approx 71.22$$

6. (c) A good guess, since the hypotenuse is 13, would be the 5-12-13 right triangle. The area is indeed 30 and the perimeter is 30. If you did not guess this,

you could use the equations $a^2 + b^2 = 13^2 = 169$ and $\frac{1}{2}ab = 30 \Rightarrow 2ab = 120$. Adding and subtracting the second equation from the first gives $a^2 + 2ab + b^2 = 169 + 120 = 289$ and $a^2 - 2ab + b^2 = 169 - 120 = 49$. Thus $(a+b)^2 = 289 \Rightarrow a + b = 17$ and $(a-b)^2 = 49 \Rightarrow a - b = 7$. These last two imply that a = 12 and b = 5.

7. (a) The simplest way to find this is to put a rectangle around the given triangle and then subtract off the areas of the right triangles that surround it. (See Figure) The area of the entire rectangle is 126, but after subtracting the areas of the 3 right triangles, whose areas are 54, 3, and 42, we are left with an area of 27.



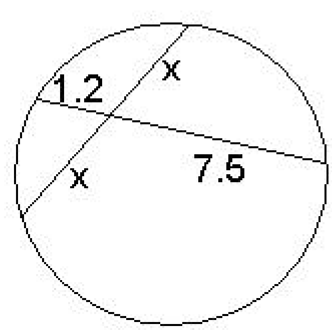
8. (e) We want the two angles, from vertical or the 12 on the clock, going clockwise, to be equal for both hands. The minute hands starts at an angle of zero and rotates at 1/60 revolutions per minute. The hour hand, which begins at 1/4th of a revolution, moves at 1/720 revolutions per minute. (It takes 12 hours times 60 minutes to make one complete revolution.) Setting these equal we have $\frac{1}{60}t = \frac{1}{4} + \frac{1}{720}t \Rightarrow 12t = 180 + t \Rightarrow 11t = 180 \Rightarrow t = 16.\overline{36}$.

9. (a) Let the sides have length x and 4.5. Then we have $4.5x = 2(2x+9)$, so $0.5x = 18 \Rightarrow x = 36$.

10. (b) The circumference of the can us $cp = 2pr$, so the radius is $r = \frac{c}{2}$. This side of the square piece is the same as the circumference. The volume is

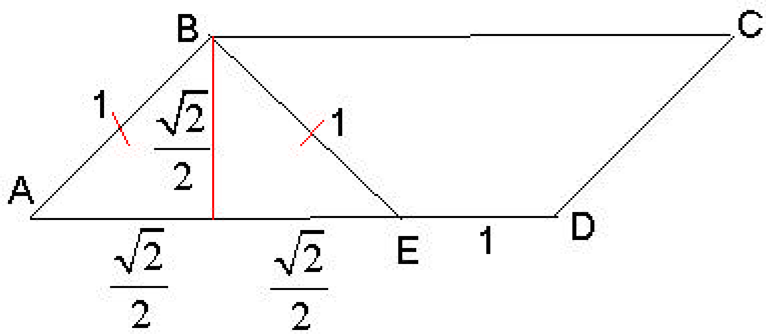
$$V = pr^2h = p\left(\frac{c}{2}\right)^2(pc) = \frac{p^2c^3}{4}.$$

11. (a) By the power of a point formula, we know that $x^2 = 1.2 \cdot 7.5 = 9 \Rightarrow x = 3$



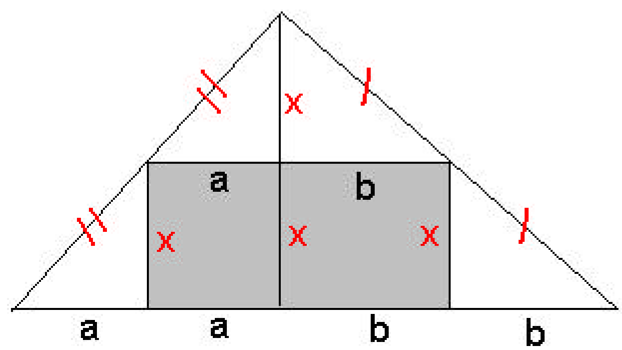
12. (d) The area of a rectangle is base times height. In the figure we see that the base is

$\sqrt{2}+1$ and the height is $\frac{\sqrt{2}}{2}$, so the area is $A = Bh = \left(\sqrt{2}+1\right)\frac{\sqrt{2}}{2} = \frac{2+\sqrt{2}}{2}$

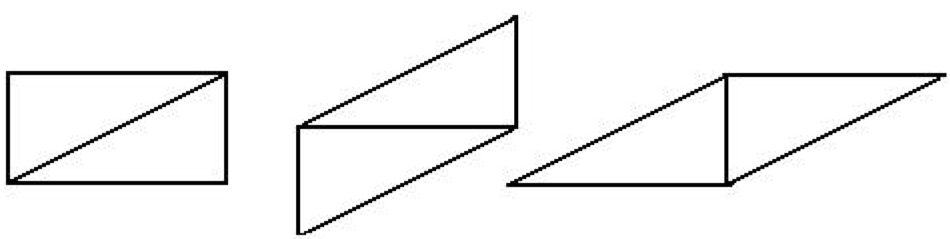


13. (c) $z + y < 7 \Rightarrow 0 < z < 7$ and $0 < y < 7$. $x + z > 10$ and $0 < z < 7 \Rightarrow x \geq 3$. $x \geq 3 \Rightarrow y \geq 3$. $y \geq 3$ and $0 < y < 7 \Rightarrow y = 3,4,5,6$.

14. (e) In the figure we see that the area of the shaded region is $x(a+b)$. The area of the entire triangle is $\frac{1}{2}(2x)(2a+2b) = 2x(a+b)$. Since the area of the shaded region is one-half the area of the entire triangle, the ratio of the shaded region to the unshaded region is 1:1.



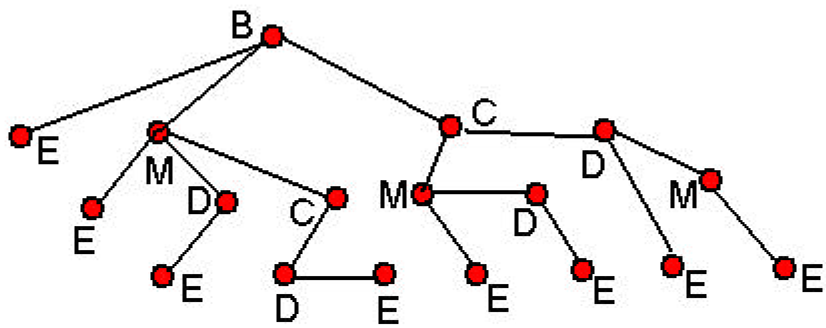
15. (c) The three configurations are shown here. In each case the area must be the sum of the two areas, since there is no overlap. The other statements are false, depending on the arrangement.



16. (a) The center of the circumscribed circle must be the same distance from each vertex. The perpendicular-bisector of one side is equidistant from two vertices, so the point of intersection of two perpendicular-bisectors is equidistant from all 3 vertices.

17. (e) The segments joining the centers will be formed by the radii, so the sides of the triangle are 3, 4, and 5 units long, making it a 3-4-5 right triangle with are 6.

18. (e) The are 2 ways to get to point B, so we can start at B and then double our answer. A tree diagram shows there are 8 ways to get from B to S without visiting any point twice.



19. (c) The radius is 7, so the circumference is $14p$. Divide this by 5 to get $2.8p$ for the arc length.

20. (b) The perimeter is $2a+2b = p \Rightarrow a+b=\frac{p}{2}$. Squaring we have

$(a+b)^2 = \left(\frac{p}{2}\right)^2 \Leftrightarrow a^2+2ab+b^2 = \frac{p^2}{4}$. But the diagonal $c^2 = a^2+b^2$, so

subtracting we get $2ab = \frac{p^2}{4} - c^2 \Rightarrow ab = \frac{p^2-4c^2}{8}$.

21. (d) The volume of the original solid is 200 cubic units. When two sides are doubled, the volume goes to 800. The original surface areas was $2\cdot 5^2 + 4\cdot(5\cdot 8) = 210$. The new surface area will be $2\cdot 10^2 + 4(10\cdot 8) = 520$. The ratio of the volumes is $\frac{520}{210} = \frac{52}{21} = 2\frac{10}{21}$.

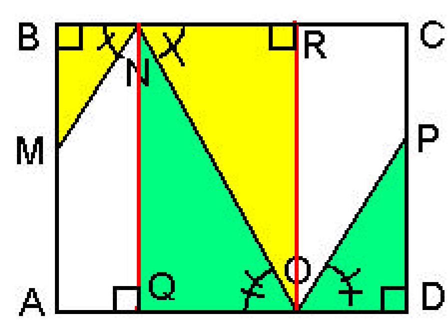
22. (b) In the figure we see that triangle MBN is similar to triangle ORN, so

$\frac{BM}{BN} = \frac{OR}{RN} \Rightarrow \frac{3}{2} = \frac{7.5}{RN} \Rightarrow RN = 5$.

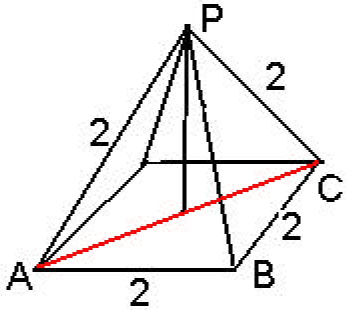
Similarly, triangle NQO is similar to triangle PDO, so

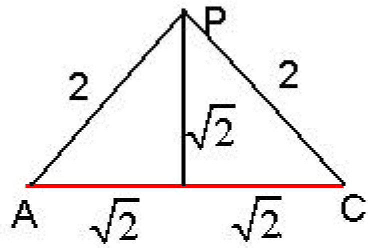
$\frac{NQ}{QO} = \frac{PD}{DO} \Rightarrow \frac{7.5}{5} = \frac{5.4}{DO} \Rightarrow DO = 3.6$.

Thus the length of the rectangle is 10.6.



23. (c) Draw in diagonal AC and look at triangle APC. The length of AC is $2\sqrt{2}$, so that makes triangle APC an isosceles right triangle and the altitude is $\sqrt{2}$.





24. (a) The old area was $9w^2$ and the perimeter was $20w$. The new area will be $18w^2$. The perimeter stays the same so the new length and width, L and W are such that $L = 10w - W$, and $(10w - W)W = 18w^2$. This is quadratic in W, so

$$W^2 - 10wW - 18w^2 = 0 \Rightarrow W = \frac{10w \pm \sqrt{100w^2 - 72w^2}}{2}$$. This simplifies to

$W = \left(5 \pm \sqrt{7}\right) \Rightarrow L = \left(5 \mp \sqrt{7}\right)$. Taking the length to be the longer side, the ratio is

$$\frac{5+\sqrt{7}}{5-\sqrt{7}}.$$

25. (e) Each of the five statements involves only three statements. We will call these C for the Car is locked, T for the Tickers are stolen, and W for the wallet is stolen. A truth table of all 8 possibilities is shown here:

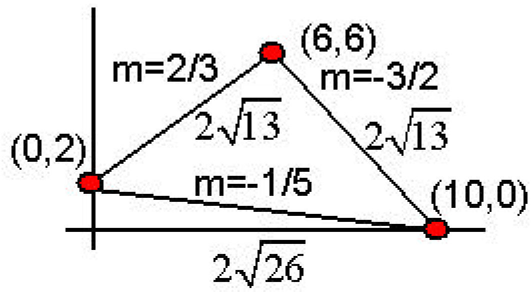
C	T	W	~C	$\sim C \to W$	~T	$W \cap \sim T$	$T \to (\sim C \to W)$
T	T	T	F	T	F	F	T
T	T	F	F	T	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	F	T
F	T	T	T	T	F	F	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	F	T	F	T
				Chris and Ann	Dan	Ed	Bo

Since we are told that exactly one of the statements is false and the rest true, this only happens in row 4, where Ed makes the false statement and the others make true ones.

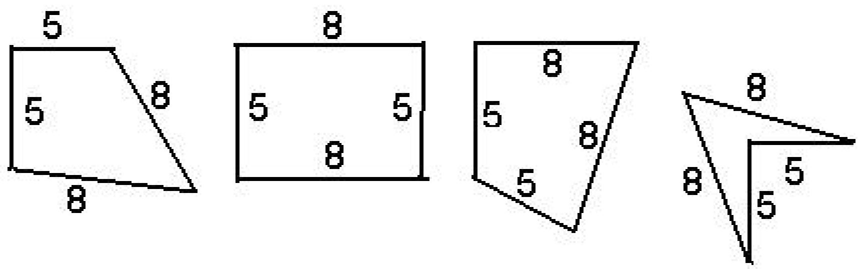
26. (d) Opposite angles in an inscribed quadrilateral must be supplementary, since the arcs they subtend form the entire circle. Thus the measure of the angle at vertex C must be 144 degrees.

27. (e) The surface area to paint is $2(8\cdot 20)+2(8\cdot 14)-2(3\cdot 7)-4(6\cdot 5)=382$. Since we need two coats, we need enough paint to cover 764 sq feet, or 1.91 gallons.

28. (d) In the figure we see the slopes and lengths of the sides of this triangle. Since two of the slopes are negative reciprocals, and since the converse of the Pythagorean theorem applies, we do have an isosceles right triangle.



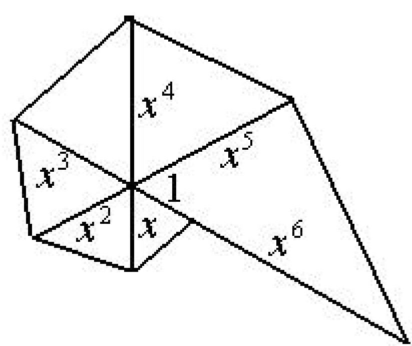
29. (d) The four possible figures are shown here. Note that the problem does not say that one of the angles of the quadrilateral has to be a right angle, only that two of the sides meet at a righ angle. The allows the one concave polygon.



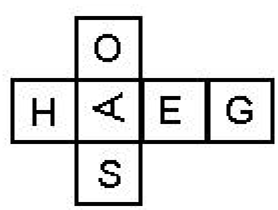
30. (c) The sides of the new triangle will be 8, 10 and 12. Thus the perimeter is 30 and the semiperimeter is 15. Using Heron's Formula, the area is $\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{15\cdot 7\cdot 5\cdot 3}=15\sqrt{7}$.

31. (e) Each angle of a regular n-sided polygon is $\dfrac{(n-2)180}{n}$, so in this case each angle is $\dfrac{(12-2)180}{12}=150$.

32. (c) Since each of the triangles are similar, ratios of corresponding sides leads us to conclude that $x^6=6$, so $x=6^{1/6}$. The areas of the triangles are $\frac{1}{2}\cdot 1\cdot x\cdot \sin(60^\circ)$ and $\frac{1}{2}x^5\cdot x^6\cdot \sin(60^\circ)$, so the ratio is of their areas is $\dfrac{\frac{1}{2}x^{11}\sin(60)}{\frac{1}{2}x\sin(60)}=x^{10}=\left(6^{1/6}\right)^{10}=6^{5/3}$.



33. (e) The cube can be flattened to look like the figure, showing that G is on the opposite side of the block from A.

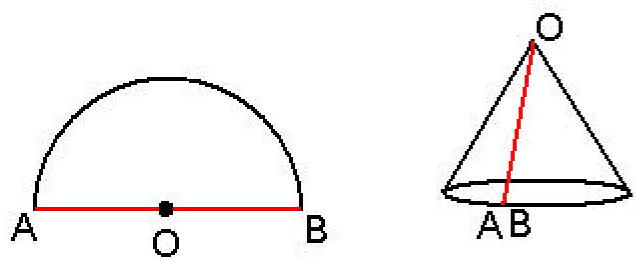


34. (c) The radius of the largest circle is $\frac{a+b}{2}$ while the radii of the two smaller circles are $\frac{a}{2}$ and $\frac{b}{2}$. Thus the area of the shaded region is $\frac{p}{2}\left(\left(\frac{a+b}{2}\right)^2-\left(\frac{a}{2}\right)^2-\left(\frac{b}{2}\right)^2\right)$. This simplifies to $\frac{p}{2}\left(\frac{a^2+2ab+b^2-a^2-b^2}{4}\right)=\frac{p\,ab}{4}$.

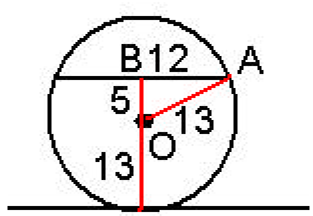
35 (a) If the diameter of the wheel is 70cm, the circumference is $70p\text{ cm}$. Dividing this into $120km \cdot 1000\frac{m}{km} \cdot 100\frac{cm}{m} = 120000000cm$ yields about 54567.4 revolutions. This is pretty close to choice a.

36. (b) The volume of a regular solid is proportional to one edge cubed. The surface area is proportional to the side squared, so the surface area is proportional to the volume raised to the two-thirds power, $SA = kV^{2/3}$. If the volume is doubled, the new surface area will be $(2V)^{2/3} = 2^{2/3}V^{2/3}$.

37. (a) As shown in the figure, the midpoint of the diameter becomes the vertex of the cone and points A and B meet on the circumference of the base of the cone. The radius OA becomes the slant height of the cone and the half-circumference of the semicircle becomes the circumference of the cone. So $6p = 2p\,r$, where r is the radius of the cone. The radius is then 3, the slant height 6, so the altitude is $\sqrt{6^2-3^2}=\sqrt{27}=3\sqrt{3}$.

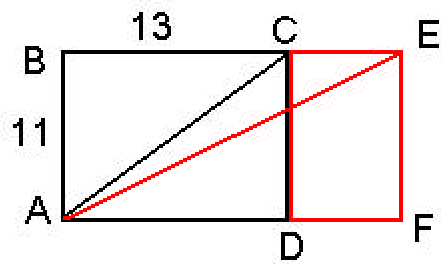


Thus the volume of the cone is $\frac{p}{3}r^2h=\frac{p}{3}\cdot 3^2\cdot 3\sqrt{3} = 9p\sqrt{3}$.



38. (d) Through the center of the circle O, drop the perpendicular to the tangent line. This will have length 18, and will include the radius of length 13 and segment OB of length 5. By the Pythagorean theorem, AB = 12 and the chord has length 24, since the segment perpendicular to the chord also bisects it.

39. (e) The original diagonal AC has length $\sqrt{11^2+13^2}=\sqrt{290}$. The new diagonal, AE has length $1.5\sqrt{290}$, so the new side of the rectangle has length $\sqrt{\left(1.5\sqrt{290}\right)^2-11^2}=\sqrt{531.50}\approx 23.054$. Thus the new area is approximately $11\cdot 23.054=253.597$.



40. (b) First, several simplifying assumptions need to be made. First, we need to assume that the 15 miles is to the center of the mountain, the point directly below the peak. Next we need to decide whether the mountain is 15 miles from the lamp or Nikki. (It is not clear from the problem statement.) Let's assume it is from the lamp. Then, using similar triangles, we have $\frac{10}{250}=\frac{h}{15\cdot 5280}\Rightarrow h=\frac{10\cdot 15\cdot 5280}{250}=3168$. Choice b is the only one even close.

If we assume that the mountain is 15 miles from where Nikki is standing we get $\frac{10}{250}=\frac{h}{15\cdot 5280+250}\Rightarrow h=\frac{10\cdot(15\cdot 5280+250)}{250}=3178$. This equals choice b.