Algebra I
State Mathematics Contest Finals, May 1, 2003

1. c $y = x-1, y = 2x+1 \Rightarrow x-1 = 2x+1 \Rightarrow -2 = x$

2. d $\left(a^{-1}-2b\right)^2 = \left(a^{-1}\right)^2 - 2\left(a^{-1}\right)\left(2b\right) + \left(2b\right)^2 = a^{-2} - 4a^{-1}b + 4b^2$
$= \frac{1}{a^2} - \frac{4b}{a} + 4b^2$

3. b Since $y = -\frac{1}{2}\left(x^2-4x-5\right) = -\frac{1}{2}(x-5)(x+1)$, we know that the highest point (the vertex) occurs when x = 2, making y = 4.5. In addition, the function is zero when x = -1 and 5.

4. a Since $f(x) = \frac{x-1}{x+1}, f(1-x) = \frac{(1-x)-1}{(1-x)+1} = \frac{-x}{2-x} = \frac{x}{x-2}$

5. e $x^3 - 6x = x^2 \Leftrightarrow x^3 - x^2 - 6x = 0 \Leftrightarrow x(x-3)(x+2) = 0$
$\Rightarrow x = 0, 3, -2$

6. b The surface area is $6s^2$ and the sum of the lengths of the 12 edges is $12s$, so $6s^2 = 12s \Rightarrow 6s^2 - 12s = 0 \Rightarrow 6s(x-2) = 0 \Rightarrow x = 2 \Rightarrow V = 2^3 = 8$

7. a One pattern that works for the first four numbers is to add 5 then take half. This would make the next term $(35+5) \div 2 = 20$ and the next term would be $(20+5) \div 2 = 12.5$. (Note. There are many other possible answers, but this is the most obvious.)

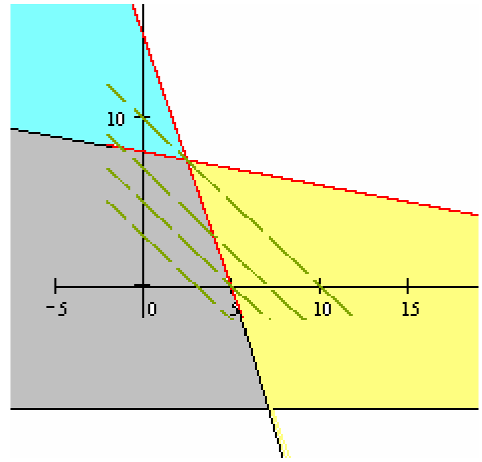
8. c Since the line intersects y = x when x = 3, y must also be 3, so the line contains the point (3,3). Thus $3(3) - 5(3) = 9 - 15 = -6$, so one equation is $3x - 5y = -6$. This is equivalent to $y = \frac{3}{5}x + \frac{6}{5}$.

9. e $\frac{x+c}{x+1} = c+1 \Rightarrow x+c = (x+1)(c+1) = xc + x + c + 1 \Rightarrow xc + 1 = 0$
$\Rightarrow x = \frac{-1}{c}$

10. b Chris's allowance in month n will be $C(n) = 10(n)$. Pat's is $P(n) = 0.10(2^{n-1})$. The values, using your calculator, would be

N	1	2	3	4	5	6	7	8	9	10	11	12
C(n)	10	20	30	40	50	60	70	80	90	100	110	120
P(n)	0.10	0.20	0.40	0.80	1.60	3.20	6.40	12.80	25.60	51.20	102.40	204.80

11. c When the two regions are graphed, we see that the point of intersection is the place where $x + y$ would be the greatest. (Lines $x + y = c$ run diagonally across the first quadrant and will have the greatest sum at the corner. This point is found by setting $8 - 0.2x = 15 - 3x \Rightarrow 40 - x = 75 - 15x$ $\Rightarrow 14x = 35 \Rightarrow x = \frac{35}{14} = \frac{5}{2}$ and $y = 15 - 3\left(\frac{5}{2}\right) = \frac{15}{2} \Rightarrow x + y = 10$



12. b Since A = B + C, A is larger than either B or C. Since D = 2B – A, $B = \frac{D+A}{2}$, so B is between A and D. So the order of these three is DBA. C = D + B – A and C = A – B implies that 2C = D, so C is smaller than D, making the order CDBA.

13. e $2x - 3 = x^2 + 2x - 7 \Rightarrow 0 = x^2 - 4 \Rightarrow x = \pm 2$. The points of intersection are (2,1) and (-2,-7) so the distance is $\sqrt{(2-(-2))^2 + (1-(-7))^2} = \sqrt{4^2 + 8^2} = \sqrt{80}$

14. e $\frac{4}{2^x} = \sqrt{8^x} \Rightarrow 4 = 2^x \cdot \left(2^{3x}\right)^{1/2} = 2^x \cdot 2^{3x/2} = 2^{5x/2} \Rightarrow 2^2 = 2^{5x/2}$ $\Rightarrow 2 = \frac{5x}{2} \Rightarrow 5x = 4 \Rightarrow x = \frac{4}{5}$

15. a Since $d = rt$, we know that $5(2.4) = 4(x) \Rightarrow 12 = 4x \Rightarrow x = 3$.

16. d Think of filling the three pairs in order. There are ${}_6C_2 = \frac{6!}{4!2!} = 15$ ways to fill the first pair. There are then ${}_4C_2 = \frac{4!}{2!2!} = 6$ ways to fill the second pair, and then the remaining two fill the last pair. Thus, there are $(15)(6) = 90$ ways to fill the pairs. There are only 3 positions for the pair of all girls (first, second or third pair) but then ${}_4C_2 = \frac{4!}{2!2!} = 6$ ways to fill the other two pairs. Thus the probability of there being two girls in a pair is $\frac{3(6)}{90} = \frac{1}{5}$.

17. e. Luther must have a total of at least $89.5(5) = 447.5$ points. His total so far is $83 + 85 + 93 + 91 = 352$ points. Thus he must score 447.5 – 352 = 95.5 points. Bus since score must be whole numbers, he must score 96.

18. a. The slopes must be the same so

$$\frac{14-5}{7-1}=\frac{c-5}{3-1} \Leftrightarrow 6(c-5)=(9)(2) \Rightarrow 6c=48 \Rightarrow c=8$$

19. a. The following linear combination of the three equations eliminates both the y and the z so that only x remains.

$$\begin{array}{l} 1(x+y+z)=1(20) \\ 2(x-y\quad)=2(6) \\ \underline{1(\quad y-z)=1(4)} \\ 3x \qquad\quad =36 \Rightarrow x=12 \end{array}$$

20. c. $y=10+3x-x^2=(5-x)(2+x)$, so the last product is positive for x-values between -2 and 5, exclusive. However, the x-values must be positive also so only x-values from 1 to 4 are allowed:

$$=(5-x)(2+x)=\big(5-(1)\big)\big(2+(1)\big)=12$$
$$=(5-x)(2+x)=\big(5-(2)\big)\big(2+(2)\big)=12$$
$$=(5-x)(2+x)=\big(5-(3)\big)\big(2+(3)\big)=10$$
$$=(5-x)(2+x)=\big(5-(4)\big)\big(2+(4)\big)=6$$

making the sum of the y-values 40.

21. b. This is a variation of the famous average speed problem. The average speed for the trip is not $\frac{55+70}{2}=62.5$ but rather $\frac{55(14)+70(11)}{14+11}=61.6$, so if the trip took 5 hours, the distance is (61.5)(6) = 308 miles.

22. c. $x^2=10+y$ and $2x=y+7$, so $y=2x-7 \Rightarrow x^2=10+\big(2x-7\big)$ and finally $x^2-2x-3=0 \Leftrightarrow (x-3)(x+1)=0$ making x equal to 3 and -1 and the corresponding y-values -1 and -9. Thus the two possible sums are 2 and -10.

23. d. Since the points are exactly 13 units apart, $13=\sqrt{(x-4)^2+(y-8)^2}$. Since $y=2.4x-1.6$, we see that

$$13^2=(x-4)^2+\big((2.4x-1.6-8\big)^2 \Leftrightarrow$$
$$169=x^2-8x+16+5.76x^2-46.08x+92.16$$
$$0=6.76x^2-54.08x-60.84$$
$$\therefore x=\frac{54.08\pm\sqrt{54.08^2-4(6.76)(-60.84)}}{2(6.76)}$$
$$=9,-1$$

but we must use the positive value, so $x = 0$ and $y=2.4(9)-1.6=20$

24. e. $(x\oplus 2)\oplus(x+2)=\left(\frac{1}{x}+\frac{1}{2}\right)\oplus(x+2)=\left(\frac{2+x}{2x}\right)\oplus(x+2)$, so

$$\frac{1}{\frac{2+x}{2x}}+\frac{1}{x+2}=\frac{2x}{x+2}+\frac{1}{x+2}=\frac{2x+1}{x+2}.$$

25. c. $2x\oplus 3=x\oplus 12\Leftrightarrow\frac{1}{2x}+\frac{1}{3}=\frac{1}{x}+\frac{1}{12}\Leftrightarrow 6+4x=12+x$, thus $3x=6\Rightarrow x=2$.

26 b. The program yields the following values for *x* and *y*:

x	0	10(0.2) + 0 = 2	9(0.2) + 2 = 3.8	8(0.2) + 3.8 = 5.4	7(0.2) + 5.4 = 6.8
y	10	9	8	7	6

In the last step *x* > *y*, so we print 6.8

27. a. The three equations are $M=C+F, M-2=3(C-2), F=C+5$. Solve this system of 3 equations to find that C = 9, F = 14, and M = 23.

28. d. Let

$$\sqrt{x^2+c}-x=c\Leftrightarrow\sqrt{x^2+c}=x+c\Rightarrow x^2+c=(x+c)^2$$

$$\Rightarrow x^2+c=x^2+2cx+c^2\Rightarrow 2cx=c-c^2\Rightarrow x=\frac{1-c}{2}$$

29. e. The slope of the given line is 4/3 so the slope of the desired line is -3/4. The point of intersection is (6,4), so the desired equation is $y-4=-0.75(x-6)\Leftrightarrow y=-0.75x+8.5$.

30. e. Since $z=\frac{2yx}{x^2+y^2}\Leftrightarrow zx^2+zy^2=2xy\Leftrightarrow zx^2-2yx+zy^2=0$. This is quadratic in x, so the Quadratic Formula yields

$$x=\frac{2y\pm\sqrt{(2y)^2-4(z)(zy^2)}}{2z}=\frac{2y\pm\sqrt{4y^2-4z^2y^2}}{2z}=\frac{2y\pm 2y\sqrt{1-z^2}}{2z}, \text{ so}$$

$$x=\frac{y\pm y\sqrt{1-z^2}}{z}.$$

31. d. The graph has been flipped over the *x*-axis, making it $-f(x)$, then moved 2 to the right, making it $-f(x-2)$, and finally moved up 4, making it $4-f(x-2)$.

32. c. Let N, D, and Q be the number of Nickels, Dimes, and Quarters, respectively. Thus $5N+10D+25Q=500$, but since the number of nickels and dimes is the same, this gives us $5N+10N+25Q=15N+25Q=500$. The possible solutions for this equation, with positive numbers of coins is

$(N,D,Q)=(30,30,2),(25,25,5),(20,20,8),(15,15,11),(10,10,14),(5,5,17)$.
The solution with the most coins is (30,30,2), or 62 coins.

33. a. $x-3=\dfrac{x+3}{x} \Leftrightarrow x(x-3)=x+3 \Leftrightarrow x^2-4x-3=0$. The two solutions for this are $x=\dfrac{4\pm\sqrt{16-4(-3)}}{2}=\dfrac{4\pm\sqrt{28}}{2}=\dfrac{4\pm 2\sqrt{7}}{2}=2\pm\sqrt{7}$. The corresponding y-values are $-1\pm\sqrt{7}$. The ordered pair $\left(2+\sqrt{7},-1+\sqrt{7}\right)$ is

$\sqrt{\left(2+\sqrt{7}\right)^2+\left(-1+\sqrt{7}\right)^2}=\sqrt{4+4\sqrt{7}+7+1-2\sqrt{7}+7}=\sqrt{19+2\sqrt{7}}$ units from the origin. The other point $\left(2-\sqrt{7},-1-\sqrt{7}\right)$ is

$\sqrt{\left(2-\sqrt{7}\right)^2+\left(-1-\sqrt{7}\right)^2}=\sqrt{4-4\sqrt{7}+7+1+2\sqrt{7}+7}=\sqrt{19-2\sqrt{7}}$ units from the origin.

34. c. Since $f(x)=\sqrt{2x-1}, f(x-2)=\sqrt{2(x-2)-1}=5$. Thus
$2(x-2)-1=25 \Rightarrow x=15$.

35. a. The following table gives the numbers of voters

Voters	Women	Men	Total
135,000	135000(0.56) = 75,600	135,000 – 75,600 = 59,400	
Democratic	75,600(0.52) = 39,312	59,400(0.475) = 28,215	39,312 + 28,215 = 67,527

36. c. For the line to be tangent to the parabola, the intersection must consist of one double root. So $x^2+2=mx-3 \Leftrightarrow x^2-mx+5=0$. To have a double root this equation must factor as $x^2-mx+5=\left(x\pm\sqrt{5}\right)^2$, so

$\left(x\pm\sqrt{5}\right)^2=x^2\pm 2\sqrt{5}x+5$, making $m=\pm 2\sqrt{5}=\pm\sqrt{20}$.

37. b. From the first equation we have $2x+6y=C+10$ and the second gives $x+2y=12$. Multiply the second by 3 to get $3x+6y=36$. Now subtract, giving $x=26-C$.

38. e. If the time is after 17:00, it has to be either 18:06 or 21:07 since the hour is 3 times the minutes. Earlier the time had to be 17:51 or 20:60 (not possible), so the time was 17:51 and 15 minutes has elapsed.

39. b. $\sqrt{2x^2-3}=2x-3 \Rightarrow 2x^2-3=\left(2x-3\right)^2=4x^2-12x+9$. So

$$2x^2-12x+12=0 \Rightarrow x=\frac{12\pm\sqrt{144-4(2)(12)}}{4}=\frac{12\pm\sqrt{48}}{4}=3\pm\sqrt{3}.$$

Checking for extraneous roots shows that

$$\sqrt{2\left(3+\sqrt{3}\right)^2-3}\ ?=?\ 2\left(3+\sqrt{3}\right)-3$$
$$\sqrt{2\left(9+6\sqrt{3}+3\right)-3}\ ?=?\ 6+2\sqrt{3}-3$$
$$\sqrt{21+12\sqrt{3}}\ ?=?\ 3+2\sqrt{3}$$
$$\sqrt{\left(3+2\sqrt{3}\right)^2}=3+2\sqrt{3}$$

which checks. However

$$\sqrt{2\left(3-\sqrt{3}\right)^2-3}\ ?=?\ 2\left(3-\sqrt{3}\right)-3$$
$$\sqrt{2\left(9-6\sqrt{3}+3\right)-3}\ ?=?\ 6-2\sqrt{3}-3$$
$$\sqrt{21-12\sqrt{3}}\ ?=?\ 3+2\sqrt{3}$$
$$\sqrt{\left(3-2\sqrt{3}\right)^2}=3-2\sqrt{3}$$

does not check since $3-2\sqrt{3}$ is negative.

40. d. By generating the first several ordered pairs, we see that

$$f(0)=0 \Rightarrow (0,0)\in f$$
$$f(1)=2f(0)+1 \Rightarrow (1,1)\in f$$
$$f(2)=2f(1)+1=3 \Rightarrow (2,3)\in f$$
$$f(3)=2f(2)+1=7 \Rightarrow (3,7)\in f$$

These ordered pairs are of the form $\left(x,2^x-1\right)$, so $f(x)=2^x-1$.