

Calculus Investigation: The Slingshot Around the Moon.

Introduction

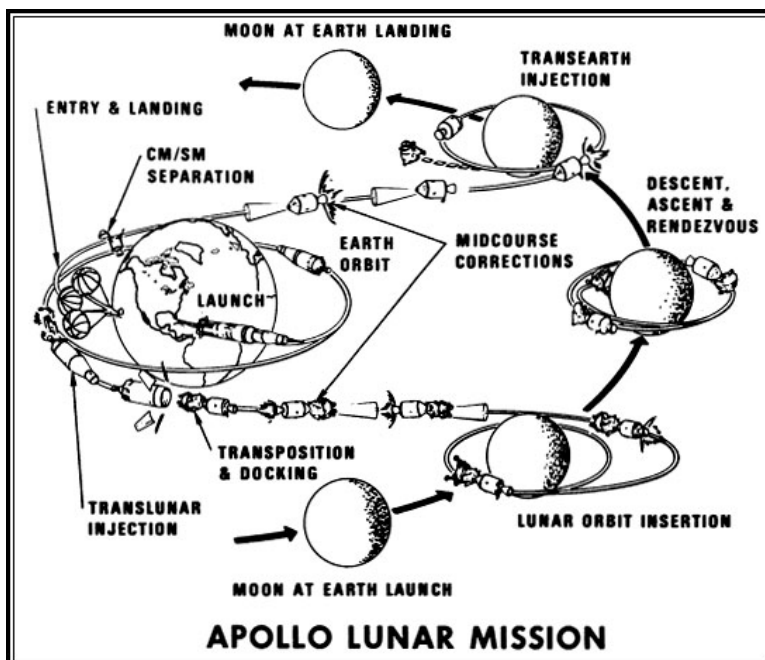
Each of the Apollo missions to the moon had a safety measure built into the spacecraft's flight path: if for some reason the rockets failed to operate properly, the path of the craft was planned such that it would "slingshot" around the moon in a *free-return trajectory*, returning to earth "powered" only by the gravitational forces of the moon and the earth.



In the Apollo 8 mission of 1968, the first to go to the moon, this flight path was adjusted slightly by three very short fuel burns measuring only a few minutes. The first burn put Apollo 8 into an elliptical orbit around the moon and the second put it into a circular orbit. Apollo 8 orbited the moon 10 times and then, after a third short burn, its trajectory was altered once more so that it could return to earth. These fuel burns, as well as the initial flight trajectory, were planned with extraordinary precision long before Apollo 8 left the earth.

Apollo 13 was launched in 1970, its crew expecting to be the fifth crew to orbit the moon and the third to land on it. The explosion of an oxygen tank on the craft 56 hours into the mission at first disappointed the crew since they knew immediately that their lunar landing would have to be aborted. Within minutes their disappointment changed to fear as they realized that their lives were in serious danger. They hadn't anywhere near enough fuel to turn around and head home. They had no choice but to continue their flight to the moon and return to earth using the free-return trajectory. Because of the escaping oxygen and debris, their flight path needed to be modified slightly throughout their trip, but essentially, their slingshot around the moon went as planned. The survival of the Apollo 13 crew would not have been possible were it not for this safety measure.

Your goal in this investigation is to create a free-return trajectory for an Apollo spacecraft—one that will send it to the moon and return the craft safely to the earth without firing thrusters after the *translunar injection*—the initial firing of the thrusters.



Gravity

The force of gravity pulling two objects towards one another is given by the following equation

$$F = G \cdot \frac{m_1 \cdot m_2}{r^2},$$

where m_1 and m_2 are the masses of the two objects, r is the distance separating the objects, and G is a constant known as the *universal gravitational constant*. Some particular constant values that may be useful in this calculus investigation are as follows:

Constant	Symbol	Value
Mass of the earth	m_E	5.97×10^{24} kg
Mass of the moon	m_M	7.35×10^{22} kg
Mass of Apollo 8	m_A	28,833 kg
Distance from earth to moon	D	384,400 km
Gravitational constant	G	8.65×10^{-13} km ³ · kg ⁻¹ · hr ⁻²
Radius of the earth	r_E	6378 km
Radius of Apollo 8's earth orbit	ro_E	6563 km
Radius of the moon	r_M	1738 km
Radius of Apollo 8's lunar orbit	ro_M	1849 km

Another fact from physics is that $F = m \cdot a$, *i.e.*, force equals mass times acceleration. What will be of more use to us in this investigation is the fact that acceleration equals force divided by mass. For example, the acceleration that Apollo 8 experiences due to the pull of the earth, is given by:

$$a_E = G \cdot \frac{m_E}{d_E^2},$$

where a_E equals the acceleration that Apollo 8 experiences due to the earth's pull, G and m_E are the gravitational constant and the mass of the earth as before, and d_E is the distance that Apollo 8 is from the earth. Similarly, the acceleration that Apollo 8 experiences due to the moon's pull is given by:

$$a_M = G \cdot \frac{m_M}{d_M^2},$$

where a_M equals the acceleration that Apollo 8 experiences due to the moon's pull, m_M is the mass of the moon as before, and d_M is the distance that Apollo 8 is from the moon.

(Through this investigation, it is recommended that you use kilometers (km) for measuring distances, kilograms (kg) for measuring mass, and hours (hr) for measuring time. To get an idea of the relative sizes of the earth and the moon and the distance between them, hold a basketball and a tennis ball 50 feet apart.)

Part I: Escaping the Earth's gravity. A one-dimensional problem.

We will first consider the simple case of a rocket being launched directly towards the moon. We want to launch it with enough force so that it will escape the earth's gravitational pull. What initial velocity is required?

To simplify the problem, we will choose a reference frame in which neither the earth nor the moon is moving. On a straight line, the earth will be located at position $x_E = 0$, and the moon will be located at position $x_M = 384,400$.

Find the point along that straight line at which the gravitational forces of the moon and the earth be exactly equal and opposite. Call that point the *zero-gravity point*.

If your spacecraft is motionless and on the earth's side of the zero-gravity point, then the earth will gradually pull your craft towards it. Likewise, if your spacecraft is motionless and on the moon's side of the zero-gravity point, then the moon will gradually haul it in. Your goal is to determine at what initial velocity your spacecraft must be launched from earth orbit towards the moon in order to get past the zero-gravity point. The differential equation describing the changing velocity of the spacecraft comes from the total acceleration due to the earth's pull and the moon's:

$$a = G \cdot \frac{m_M}{d_M^2} - G \cdot \frac{m_E}{d_E^2}.$$

If we let $x = d_E$, which is equivalent to letting x be the position of the spacecraft along the earth-moon line, then we have

$$x + d_M = D, \text{ or}$$

$$d_M = D - x.$$

Finally, that produces the second-order differential equation

$$\frac{d^2x}{dt^2} = G \cdot \frac{m_M}{(D-x)^2} - G \cdot \frac{m_E}{x^2}.$$

Notice that the only variables in this equation are $\frac{d^2x}{dt^2}$ and x —everything else is constant.

This differential equation cannot easily be solved analytically, but numerical solutions can be found using Euler's method. Starting at $x_0 = 6563$ km (the radius of Apollo 8's earth orbit), examine what happens to your Apollo spacecraft when the initial velocity is 20,000 km/hr. Use steps of size $\Delta t = 0.001$ hours and look at the first 0.6 hours. Make graphs of acceleration versus time, velocity versus time, and position versus time for the first 0.6 hours after translunar injection. Is there enough information in these graphs to determine whether 20,000 km/hr is a sufficient initial velocity for the spacecraft to escape the earth's gravitational pull?

Now examine what happens during the first full hour of the flight, still using $\Delta t = 0.001$ hours. You may notice something strange going on. In fact, 20,000 km/hr is *not* sufficient to escape earth's gravitational pull. Somewhere around $t = 0.25$ hours, the craft reverses direction and begins to head back towards the earth. It crashes into the earth around $t = 0.5$ hours. (You should verify this!) However, the equations don't "know" this, so it may be that you have Euler's method cranking away long after the spacecraft has crashed into the earth's surface.

After implementing that into your Euler equations, you may be able, using trial and error, to estimate the minimal initial velocity required for your spacecraft to escape earth's pull.

With these tools all in place, you should be able to get your spaceship to the moon. Before continuing to the next part, be sure you can answer the questions and produce the graphs below:

- Make graphs for your spacecraft of acceleration *versus* time, velocity *versus* time, and distance from the earth *versus* time.
- Estimate the minimal initial velocity required to get a spacecraft to escape the earth's gravitational sphere of influence (and pass the zero-gravity point).
- If a spacecraft is given that initial velocity while in earth's orbit and heads straight towards the moon, how long will it take it to get there?
- By how much can the duration of the trip be decreased by launching at a faster speed than that which is minimally required?
- How fast is the spacecraft going when it crashes onto the surface of the moon?

Part II: The slingshot around the Moon. A two-dimensional problem.

Now that you have successfully gotten a spacecraft to crash-land on the moon, you need to design a flight path that will get it back home again. The earth and the moon will be located at the same points as before along a horizontal line, but now a second "vertical" dimension will be added. Thus, the earth is centered at the origin (0,0) and the moon is centered at the point whose coordinates are (384400,0). Your Apollo spacecraft will be located at the coordinates (x,y), where x and y are variables.

In writing the Euler equations that will generate the flight path of your spacecraft, you will need to separate the acceleration, velocity, and position *vectors* of your craft into their x- and y-*components*. Then the Euler equations can be implemented with x- and y-components receiving separate treatment. For example, your initial conditions may look something like this:

$$\begin{aligned}
 t_0 &= 0 \\
 x_0 &= 6563 \\
 vx_0 &= 30000 \\
 y_0 &= 0 \\
 vy_0 &= -10000,
 \end{aligned}$$

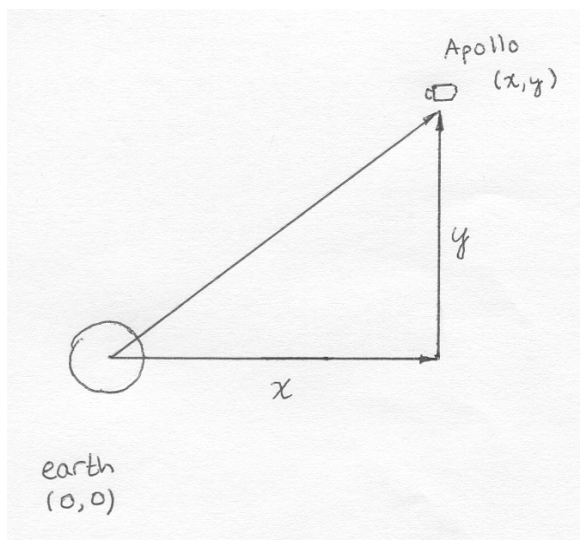
in which vx_0 stands for the x-component of the initial velocity and vy_0 stands for the y-component of the initial velocity. The (arbitrarily chosen) values given in the example above correspond to a translunar injection that occurs when the spacecraft is located in earth's orbit at

the point closest to the moon and that gives the spacecraft a velocity that has “horizontal” component equal to 30,000 km/hr and “vertical” component equal to $-10,000$ km/hr, as shown in the diagram below.

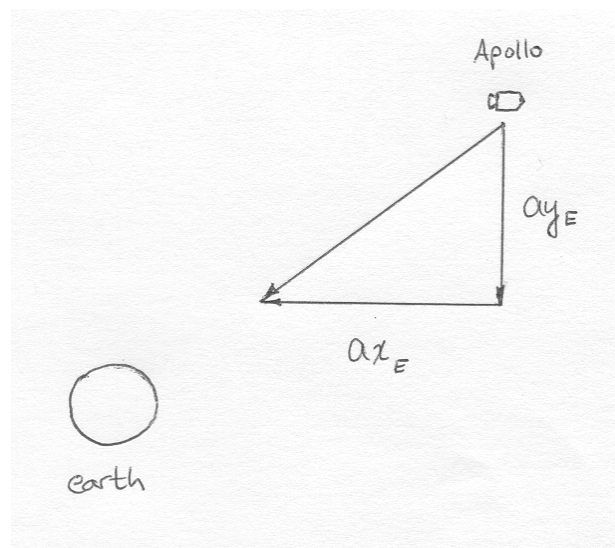


The only difficult step that remains before putting everything together is determining the x - and y - components of the spacecraft’s acceleration when the craft is located at the general coordinates (x,y) . To do this, we will use the fact that the acceleration vector of the spacecraft due to the influence of the earth or moon points in *exactly the opposite direction* as the displacement vector from the planet to the spacecraft. Consider the two diagrams below. The one on the left shows the *displacement* vector from the earth to Apollo, broken down into its x - and y - components. The diagram on the right shows Apollo’s acceleration vector, also broken down into its x - and y -components. Since the vectors point in exactly opposite directions, their components are proportional.

Displacement



Acceleration



Complete the diagrams above by writing expressions for the total distance from the earth to Apollo (in the diagram on the left) and for the total acceleration of Apollo due to the earth's pull (in the diagram on the right). These expressions should contain only known constants and the variables x and y . Then, using proportional triangles, write expressions for the x - and y -components of acceleration (called a_x and a_y in the drawing) using only known constants and the variables x and y . Pay careful attention to the *sign* (+/-) of the acceleration components. If x is positive, then a_x should be negative, *etc.*

Once you have done this, determine the x - and y -components of acceleration due to the pull of the moon.

The sum of the acceleration due to the earth and the acceleration due to the moon is equal to the total acceleration, and as before, x - and y -components may be added separately.

You now should have all the pieces you need to develop your flight path that will slingshot your spacecraft around the moon. Your goal is to begin somewhere in earth orbit and determine a *direction* and an *initial velocity* for your translunar injection such that, with no further firing of thrusters, the spacecraft will go around the moon and then come back. Some questions you may wish to consider and graphs you may wish to produce as you work are:

- How long does the round-trip voyage take?
- Make graphs of total acceleration *versus* time, total velocity *versus* time, and total distance from the earth *versus* time for your flight path.
- Make a graph of the flight path itself in two dimensions, showing the locations of the earth and the moon. Make points on the path to show where the spacecraft is after one day, two days, *etc.*
- How sensitive is the flight path to changes in launch velocity or launch direction?
- Are there multiple solutions to this problem?

This is a challenging investigation. As you work, keep in mind that although the problem as it is presented here is simplified, you are studying the essentially the same problem and are doing the same calculations that NASA scientists and engineers did in the 1960's as they attempted to fulfill President Kennedy's mandate to put a man on the moon before 1970. If this appeals to you, then you may want to explore even more difficult problems, some of which require some understanding of physics.

- How much force is required to accelerate the spacecraft from an earth-orbital velocity to the velocity you chose for your flight path? How much rocket fuel would be required to accelerate Apollo 8 to that new velocity?
- Plan a fuel burn near the moon that will take the spacecraft out of its free-return trajectory and put it into lunar orbit. Then plan another fuel burn that will put it back into a free-return trajectory.
- 56 hours into Apollo 13's mission, an oxygen tank exploded in the service module and spewed oxygen and debris into space. Suppose that happened on the mission you planned. How much would that alter the flight path? Would you still be able to return to the earth.