

Calculus III Topics Exam Review Solutions
June 2008

1. a. $\int_0^{\pi} \cos^2 x dx = \int_0^{\pi} \left(\frac{1}{2} \cos(2x) + \frac{1}{2} \right) dx = \frac{1}{4} \sin(2x) + \frac{1}{2} x \Big|_0^{\pi} = \frac{\pi}{2}$

b. $\int x\sqrt{1+kx} dx$

$f = x \quad g' = (1+kx)^{\frac{1}{2}}$

$f' = 1 \quad g = \frac{2}{3k} (1+kx)^{\frac{3}{2}}$

So $\int x\sqrt{1+kx} dx =$

$\frac{2}{3k} x(1+kx)^{\frac{3}{2}} - \frac{2}{3k} \int (1+kx)^{\frac{3}{2}} dx =$

$\frac{2}{3k} x(1+kx)^{\frac{3}{2}} - \frac{2}{3k} \frac{1}{k} \frac{(1+kx)^{\frac{5}{2}}}{\frac{5}{2}} + C =$

$\frac{2}{3k} x(1+kx)^{\frac{3}{2}} - \frac{4}{15k^2} (1+kx)^{\frac{5}{2}} + C$

c. $\int t^2 e^{kt} dt$

$f = t^2 \quad dg = e^{kt} dt$

$df = 2t dt \quad g = \frac{1}{k} e^{kt}$

So $\int t^2 e^{kt} dt =$

$\frac{t^2}{k} e^{kt} - \frac{2}{k} \int t e^{kt} dt$

$f = t \quad dg = e^{kt} dt$

$df = dt \quad g = \frac{1}{k} e^{kt}$

And so $\int t^2 e^{kt} dt = \frac{t^2}{k} e^{kt} - \frac{2}{k} \left[\frac{t}{k} e^{kt} - \frac{1}{k} \int e^{kt} dt \right]$

$= \frac{t^2}{k} e^{kt} - \frac{2t}{k^2} e^{kt} + \frac{2}{k^3} e^{kt} + C$

d. $\int x \ln(x) dx$

Let $f = \ln(x)$ and $dg = x dx$. Then $df = \frac{1}{x} dx$ and $g = \frac{1}{2} x^2$, and so $\int x \ln(x) dx =$

$\frac{1}{2} \ln(x) x^2 - \int \frac{1}{2} x^2 \frac{1}{x} dx = \frac{1}{2} x^2 \ln(x) - \frac{1}{4} x^2 + C$

e. $\int e^{2x} \sin x dx$

Let $f = e^{2x}$ and $g = \sin x$, so

$f' = 2e^{2x}$ and $g = -\cos x$

Then $\int e^{2x} \sin x dx = -e^{2x} \cos x - \int 2e^{2x} (-\cos x) dx = -e^{2x} \cos x + \int 2e^{2x} \cos x dx$

Let $f_1 = 2e^{2x}$ and $g_1' = \cos x$, and so

$f_1' = 4e^{2x}$ and $g_1 = \sin x$

Giving $\int e^{2x} \sin x dx = -e^{2x} \cos x + 2e^{2x} \sin x - \int 4e^{2x} \sin x dx$

Now: $5 \int e^{2x} \sin x dx = -e^{2x} \cos x + 2e^{2x} \sin x$

And so $\int e^{2x} \sin x dx = \frac{1}{5} (-e^{2x} \cos x + 2e^{2x} \sin x) + C$

2. a. rotate about $x = 4$:

$$\int_{-3\sqrt{3}}^0 \pi \left(4 - (\sqrt[3]{y} + 4) \right)^2 - \pi \left(4 - \left(\frac{y}{3} + 4 \right) \right)^2 dy + \int_0^{3\sqrt{3}} \pi \left((\sqrt[3]{y} + 4) - 4 \right)^2 - \pi \left(\left(\frac{y}{3} + 4 \right) - 4 \right)^2 dy =$$

$$2\pi \int_0^{3\sqrt{3}} \left(\sqrt[3]{y} \right)^2 - \left(\frac{y}{3} \right)^2 dy = 2\pi \int_0^{3\sqrt{3}} y^{2/3} - \frac{1}{9} y^2 dy = \frac{24\pi\sqrt{3}}{5} \approx 26.1187$$

b. rotate about $y = 0$:

$$\int_{4-\sqrt{3}}^4 \pi (3x-12)^2 - \pi \left((x-4)^3 \right)^2 dx + \int_4^{4+\sqrt{3}} \pi (3x-12)^2 - \pi \left((x-4)^3 \right)^2 dx =$$

$$2\pi \int_4^{4+\sqrt{3}} 9(x-4)^2 - (x-4)^6 dx = \frac{72\pi\sqrt{3}}{7}$$

3. $e^x \approx \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{-x^2} \approx \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}. \text{ So, } e^{-x^2} \approx 1 - x^2. \quad \frac{1}{\sqrt[4]{e}} = e^{-1/4} = e^{-x^2} \Big|_{x=0.5} \approx 1 - (0.5)^2 = 0.75.$$

4. $\sin(x) \approx \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin\left(\frac{1}{2}x^2\right) \approx \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}x^2\right)^{2n+1}}{(2n+1)!}$$

$$\sin\left(\frac{1}{2}x^2\right) \approx \frac{1}{2}x^2 - \frac{1}{48}x^6$$

$$\int_0^1 \sin\left(\frac{x^2}{2}\right) dx \approx \int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{48}x^6\right) dx = \left. \frac{x^3}{6} - \frac{x^7}{336} \right|_0^1 = \frac{1}{6} - \frac{1}{336} = \frac{55}{336} \approx 0.16369$$

5. $\int_1^{\infty} \pi \left(\frac{1}{x} \right)^2 dx = \lim_{n \rightarrow \infty} \int_1^n \pi x^{-2} dx = \lim_{n \rightarrow \infty} -\pi x^{-1} \Big|_1^n = \lim_{n \rightarrow \infty} \frac{-\pi}{n} + \frac{\pi}{1} = \pi$

6. $\int_0^4 (2\sqrt{y})^2 dy = \int_0^4 4y dy = 2y^2 \Big|_0^4 = 32$

7. $\int_0^1 \sqrt{1 + \left(\frac{1}{2\sqrt{x}} \right)^2} dx \approx 1.478$. Using the distance formula, the distance between (0,0) and (1,1) is 1.414.

$$\int_9^{25} \sqrt{1 + \left(\frac{1}{2\sqrt{x}} \right)^2} dx \approx 16.127$$
. Using the distance formula, the distance between (9,3) and (25,5) is 16.125.

8. a. Use the Comparison Test. Since $\frac{\sin(n)}{n^3 + n^2 + n + 1 + \cos(n)} < \frac{1}{n^3}$ and $\sum \frac{1}{n^3}$ converges by the integral test,

$$\sum \frac{\sin(n)}{n^3 + n^2 + n + 1 + \cos(n)} \text{ converges.}$$

b. Use the Divergence Test. Since $\lim_{n \rightarrow \infty} \left| \frac{7n^5 - 9n^3 + 11n}{\pi n^5 + 9n^4 + 13n^2} \right| = \frac{7}{\pi} \neq 0$, the series diverges.

c. Use the Alternating Series Test. Since $\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = 0$ and $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$ for all n , the series converges.

d. Use the Ratio Test. Since $\lim_{n \rightarrow \infty} \left| \frac{\frac{3^{2(n+1)}}{((n+1)+2)!}}{\frac{3^{2n}}{(n+2)!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^2}{n+3} \right| = 0 < 1$, the series converges.

9. a. Use the Ratio Test to find the interval of convergence.

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} (x-1)^{n+1} / (n+1)}{(-1)^{n+1} (x-1)^n / n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-1)n}{n+1} \right| = |x-1| < 1$$

$|x-1| < 1 \Rightarrow 0 < x < 2$. However, when $x = 0$ the series is the opposite of the harmonic series so it does not converge. And when $x = 2$ it is the alternating harmonic series so it does converge. Thus the interval of convergence is $0 < x \leq 2$ or $(0, 2]$.

b. Use the Ratio Test to find the interval of convergence.

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 2^{n+1} (x+3)^{n+1} / (n+1)!}{(-1)^n 2^n (x+3)^n / (n)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{2(x+3)}{n+1} \right| = 0$$

The limit is less than 1 for all x -values, so the interval of convergence is $(-\infty, \infty)$.

c. Use the Ratio Test to find the interval of convergence.

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1} (x-4)^{n+1} / 3^{n+1}}{n^n (x-4)^n / 3^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-4)}{3} \cdot \frac{(n+1)^{n+1}}{n^n} \right|. \text{ Since}$$

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1}}{n^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^n (n+1)}{n^n} \right| = \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n} \right)^n \cdot (n+1) \right| = e \cdot \lim_{n \rightarrow \infty} |(n+1)| = \infty, \text{ we know that}$$

$\lim_{n \rightarrow \infty} \left| \frac{(x-4)}{3} \cdot \frac{(n+1)^{n+1}}{n^n} \right|$ is less than 1 only for $x = 4$. The interval of convergence consists of a single number, 4.

d. Use the Ratio Test to find the interval of convergence.

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 3^{n+1} (x+5)^{n+1}}{(-1)^n 3^n (x+5)^n} \right| = \lim_{n \rightarrow \infty} |3(x+5)| < 1$$

$|3(x+5)| < 1 \Rightarrow -5\frac{1}{3} < x < -4\frac{2}{3}$. When $x = -5\frac{1}{3}$ every term in the series is equal to one so it does not converge. And when $x = -4\frac{2}{3}$ the series becomes $\sum (-1)^n$ so it does not converge. Thus the interval of convergence is $-5\frac{1}{3} < x < -4\frac{2}{3}$.

10. $\sqrt{x} \approx 1 + \frac{1}{2}(x-1) - \frac{1}{4} \frac{(x-1)^2}{2!} + \frac{3}{8} \frac{(x-1)^3}{3!} - \frac{15}{16} \frac{(x-1)^4}{4!}$

11. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$, and therefore $e^{-2x} = 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \dots + \frac{(-2x)^n}{n!} = \sum \frac{(-1)^n 2^n x^n}{n!}$.

Using the ratio test to find where the series converges:

$$\lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(2x)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2x}{n+1} \right| = 0 < 1. \text{ Therefore, the series converges for all real numbers.}$$

12. i. a) is a set of functions whose derivative is $f(x)$, b) is a numerical value (the net change in $F(x)$ from $x = 1$ to $x = 3$ where $F'(x) = f(x)$, and c) is the unique function whose derivative is $f(x)$ and that contains the point $(0,0)$. You should be able to elaborate on each of these and give more specific information.

ii. $\lim_{\Delta x \rightarrow 0} \sum_{k=0}^{n-1} f(x_k) \Delta x$ is equal to the integral in part b), if the limit exists. It also must be true that

$$\Delta x = \frac{3-1}{n}, x_0 = 1, x_n = x_{n-1} + \Delta x \text{ and } x_n = 3.$$

13. $F(x) = \int_{\pi}^x \tan(t^2) dt + e$

14. $F'(x) = \sec^5(x^3) \cdot 3x^2 - \sec^5(3^x) \cdot \ln(3) \cdot 3^x$

15. a. If a function $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) there exists a value c in that interval such

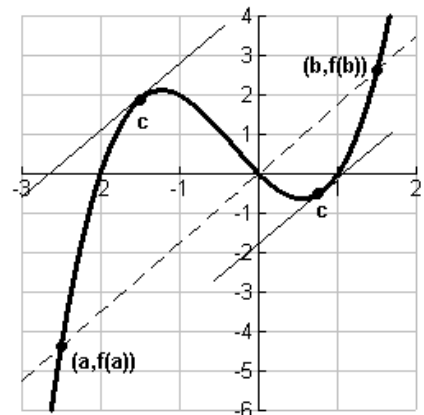
$$\text{that } f'(c) = \frac{f(b) - f(a)}{b - a}.$$

c. $f(1) = 2$ and $f(2) = \frac{5}{2}$, so $\frac{f(b) - f(a)}{b - a} = \frac{1}{2}$.

Differentiating gives $f'(x) = 1 - \frac{1}{x^2}$; to get the desired c ,

solve $f'(c) = \frac{1}{2}$. Doing so we see that $c = \sqrt{2}$. Observe that

$1 < \sqrt{2} < 2$, as was claimed by the Mean Value Theorem.



16. volume = $\int_0^4 \frac{1}{2}(\sqrt{4-x})^2 dx = 4$

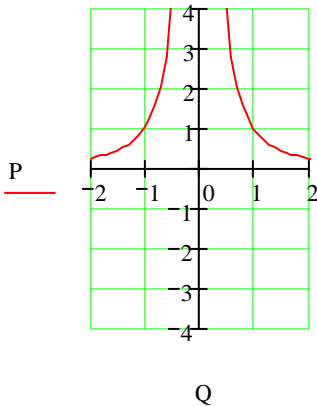
17. a. probability = $\int_{14}^{\infty} 0.3e^{-0.3x} dx \approx 0.015$ b. 150 free lunches c. mean = $\int_0^{\infty} x \cdot 0.3e^{-0.3x} dx \approx 3.33$

18. a. area must be 1: $\int_1^{\infty} \frac{3}{x^4} dx = -x^{-3} \Big|_1^{\infty} = 1$ b. mean = $\int_1^{\infty} x \cdot \frac{3}{x^4} dx = \int_1^{\infty} 3x^{-3} dx = -\frac{3}{2}x^{-2} \Big|_1^{\infty} = \frac{3}{2}$

c. median: find M such that $\int_1^M \frac{3}{x^4} dx = 0.5$; $M = \sqrt[3]{2} \approx 1.26$. On average, students visit the dentist every

1.5 years. For half of the students the time between visits is 1.26 years or less; for the other half of the students the time between visits is 1.26 years or longer. The mean is larger than the median because the large values of the random variable can be very large---there is no upper limit---while the smallest value of the random variable is zero. The large values of the random variable do not pull up the value of the median but they do pull up the value of the mean.

19. probability $PQ^2 > 1 =$ probability $P > \frac{1}{Q^2} = \frac{2 \cdot \int_{1/2}^2 4 - \frac{1}{x^2} dx}{32} = \frac{9}{32}$



20. average value = $\frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} \sin(2t)e^{1-\cos(2t)} dt = \frac{1}{2\pi} \left(\frac{1}{2} e^{1-\cos(2t)} \right) \Big|_{-\pi}^{\pi} = \frac{1}{2\pi} \left[\frac{1}{2} e^0 - \frac{1}{2} e^0 \right] = 0$

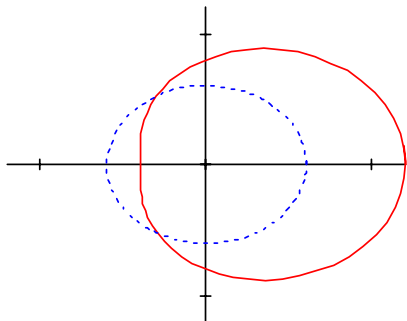
21. $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{(3+8\sin\theta)\cos\theta + 8\cos\theta\sin\theta}{(3+8\sin\theta)(-\sin\theta) + 8\cos\theta\cos\theta}$; the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{6}$ is $\frac{11\sqrt{3}}{5}$. When

$\theta = \frac{\pi}{6}$, $y = r \sin \theta = \frac{7}{2}$ and $x = r \cos \theta = \frac{7\sqrt{3}}{2}$. The equation of tangent line is

$y - \frac{7}{2} = \frac{11\sqrt{3}}{5} \left(x - \frac{7\sqrt{3}}{2} \right)$ or $y = \frac{11\sqrt{3}}{5}x - \frac{196}{10} = \frac{11\sqrt{3}}{5}x - \frac{196}{10} = \frac{11\sqrt{3}}{5}x - \frac{98}{5}$.

$$22. \text{ area} = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2} (4 + 8 \sin \theta)^2 d\theta = 16\pi - 24\sqrt{3} \approx 8.69626$$

23.



$$\text{Area} = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2} \cdot 3^2 - \frac{1}{2} (4 + 2 \cos \theta)^2 d\theta = \frac{15\sqrt{3}}{2} - 3\pi \approx 3.5656$$