

1. a. Using a trig identity:  $\int_0^{\pi} \cos^2 x dx = \int_0^{\pi} \left( \frac{1}{2} \cos(2x) + \frac{1}{2} \right) dx = \frac{1}{4} \sin(2x) + \frac{1}{2} x \Big|_0^{\pi} = \frac{\pi}{2}$

OR

Using integration by parts:

$$u = \cos x \quad \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = -\sin x \quad v = \sin x$$

$$\int_0^{\pi} \cos^2 x dx = \cos x \sin x \Big|_0^{\pi} + \int_0^{\pi} \sin^2 x dx = \cos x \sin x \Big|_0^{\pi} + \int_0^{\pi} (1 - \cos^2 x) dx$$

$$2 \int_0^{\pi} \cos^2 x dx = \cos x \sin x \Big|_0^{\pi} + \int_0^{\pi} dx = \cos x \sin x + x \Big|_0^{\pi}$$

$$\int_0^{\pi} \cos^2 x dx = \frac{\cos x \sin x + x}{2} \Big|_0^{\pi} = \frac{\pi}{2}$$

b.  $\int x\sqrt{1+kx} dx$

$$u = x \quad \frac{dv}{dx} = (1+kx)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 1 \quad v = \frac{1}{k} \frac{2(1+kx)^{\frac{3}{2}}}{3}$$

So  $\int x\sqrt{1+kx} dx =$

$$\frac{2}{3k} x(1+kx)^{\frac{3}{2}} - \frac{2}{3k} \int (1+kx)^{\frac{3}{2}} dx = \frac{2}{3k} x(1+kx)^{\frac{3}{2}} - \frac{2}{3k} \frac{1}{k} \frac{(1+kx)^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= \frac{2}{3k} x(1+kx)^{\frac{3}{2}} - \frac{4}{15k^2} (1+kx)^{\frac{5}{2}} + C$$

c. For  $\int_0^1 \frac{1}{x^2-4} dx$ , use partial fraction decomp:

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

Solve to get:

$$A = \frac{1}{4}, \quad B = -\frac{1}{4}$$

$$\begin{aligned} \int_0^1 \frac{1}{x^2-4} dx &= \int_0^1 \left( \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2} \right) dx \\ &= \frac{1}{4} \ln|x-2| + \frac{1}{4} \ln|x+2| \Big|_0^1 \\ &= -\frac{1}{4} \ln 3 \end{aligned}$$

$$\mathbf{d.} \quad \int_1^{\infty} \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx \quad \text{Let } u = \ln x \text{ and } \frac{du}{dx} = \frac{1}{x}.$$

$$\text{Then } \int u \frac{du}{dx} dx = \frac{u^2}{2} + C$$

$$\lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \left. \frac{(\ln x)^2}{2} \right|_1^t = \lim_{t \rightarrow \infty} \frac{(\ln t)^2}{2} - \frac{(\ln 1)^2}{2} = \infty \text{ so the integral diverges}$$

$$\mathbf{e.} \quad \int x \ln(x) dx$$

Let  $u = \ln(x)$  and  $\frac{dv}{dx} = x$ . Then  $\frac{du}{dx} = \frac{1}{x}$  and  $v = \frac{1}{2}x^2$ , and so

$$\int x \ln(x) dx = \frac{1}{2} \ln(x) x^2 - \int \frac{1}{2} x^2 \frac{1}{x} dx = \frac{1}{2} x^2 \ln(x) - \frac{1}{4} x^2 + C$$

$$\mathbf{f} \quad \int e^{2x} \sin x dx$$

Let  $u = e^{2x}$  and  $\frac{dv}{dx} = \sin x$ , so

$$\frac{du}{dx} = 2e^{2x} \text{ and } v = -\cos x$$

$$\text{Then } \int e^{2x} \sin x dx = -e^{2x} \cos x - \int 2e^{2x} (-\cos x) dx = -e^{2x} \cos x + \int 2e^{2x} \cos x dx$$

Let  $u = 2e^{2x}$  and  $\frac{dv}{dx} = \cos x$ , and so

$$\frac{du}{dx} = 4e^{2x} \text{ and } v = \sin x$$

$$\text{Giving } \int e^{2x} \sin x dx = -e^{2x} \cos x + 2e^{2x} \sin x - \int 4e^{2x} \sin x dx$$

$$\text{Now : } 5 \int e^{2x} \sin x dx = -e^{2x} \cos x + 2e^{2x} \sin x$$

$$\text{And so } \int e^{2x} \sin x dx = \frac{1}{5} (-e^{2x} \cos x + 2e^{2x} \sin x) + C$$

$$\mathbf{g} \quad \int t^2 e^{kt} dt$$

$$u = t^2 \quad \frac{dv}{dt} = e^{kt}$$

$$\frac{du}{dt} = 2t \quad v = \frac{1}{k} e^{kt}$$

$$\text{So } \int t^2 e^{kt} dt = \frac{t^2}{k} e^{kt} - \frac{2}{k} \int t e^{kt} dt$$

$$u = t \quad \frac{dv}{dt} = e^{kt}$$

$$\frac{du}{dt} = 1 \quad v = \frac{1}{k} e^{kt}$$

$$\text{And so } \int t^2 e^{kt} dt = \frac{t^2}{k} e^{kt} - \frac{2}{k} \left[ \frac{t}{k} e^{kt} - \frac{1}{k} \int e^{kt} dt \right]$$

$$\text{h. } \int_3^\infty \frac{dx}{x^2} = \lim_{t \rightarrow \infty} \int_3^t \frac{dx}{x^2} = \lim_{t \rightarrow \infty} \left. \frac{-1}{x} \right|_3^t = \lim_{t \rightarrow \infty} \left( \frac{-1}{t} + \frac{1}{3} \right) = \frac{1}{3}$$

$$\text{i. } \int_0^4 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^4 \frac{1}{(x-1)^2} dx = \lim_{k \rightarrow 1^+} \left. \frac{-1}{x-1} \right|_0^k + \lim_{k \rightarrow 1^-} \left. \frac{-1}{x-1} \right|_k^4$$

$$\text{Note: } \lim_{k \rightarrow 1^+} \frac{-1}{k-1} = -\infty \quad \text{so the integral diverges}$$

$$2. \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}x^2\right)^{2n+1}}{(2n+1)!} \approx \frac{1}{2}x^2 - \frac{1}{48}x^6$$

$$\int_0^1 \sin\left(\frac{x^2}{2}\right) dx \approx \int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{48}x^6\right) dx = \left. \frac{x^3}{6} - \frac{x^7}{336} \right|_0^1 = \frac{1}{6} - \frac{1}{336} = \frac{55}{336} \approx 0.16369$$

$$3. \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \lim_{n \rightarrow \infty} \int_1^n \pi x^{-2} dx = \lim_{n \rightarrow \infty} \left. -\pi x^{-1} \right|_1^n = \lim_{n \rightarrow \infty} \left( \frac{-\pi}{n} + \frac{\pi}{1} \right) = \pi$$

$$4. \int_0^4 (2\sqrt{y})^2 dy = \int_0^4 4y dy = 2y^2 \Big|_0^4 = 32$$

$$5. \int_0^1 \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx \approx 1.478. \text{ Using the distance formula, the distance between } (0,0) \text{ and } (1,1) \text{ is } 1.414.$$

$$\int_9^{25} \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx \approx 16.127. \text{ Using the distance formula, the distance between } (9,3) \text{ and } (25,5) \text{ is } 16.125.$$

$$6. \text{ a. } \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{1}{4} \left(\frac{-3}{4}\right)^{n-1}, \text{ a geometric series with } |r| = \frac{3}{4} < 1 \text{ so it converges. (Or use the Ratio test.)}$$

$$\text{b. Use the Ratio Test. Since } \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1)^2}{5^{n+2}} \cdot \frac{5^{n+1}}{(-1)^n n^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{-(n+1)^2}{5n^2} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{5n^2} = \frac{1}{5} < 1,$$

series converges.

c. Use integral test:  $\lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln x)^3} dx = \lim_{b \rightarrow \infty} \frac{1}{(\ln b)^2} - \frac{1}{(\ln 2)^2} = -\frac{1}{(\ln 2)^2}$  so series also converges.

d. Use the Alternating Series Test. Since  $\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = 0$  and  $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$  for all n, the series converges.

e. Use the Ratio Test. Since  $\lim_{n \rightarrow \infty} \left| \frac{\frac{3^{2(n+1)}}{((n+1)+2)!}}{\frac{3^{2n}}{(n+2)!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^2}{n+3} \right| = 0 < 1$ , the series converges.

7. a. Use the Ratio Test to find the interval of convergence.

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} (x-1)^{n+1} / (n+1)}{(-1)^{n+1} (x-1)^n / n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-1)n}{n+1} \right| = |x-1| < 1$$

$|x-1| < 1 \Rightarrow 0 < x < 2$ . However, when  $x = 0$  the series is the negative of the harmonic series so it does not converge. And when  $x = 2$  it is the alternating harmonic series so it does converge. Thus the interval of convergence is  $0 < x \leq 2$  or  $(0, 2]$ .

b. Use the Ratio Test to find the interval of convergence.

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 2^{n+1} (x+3)^{n+1} / (n+1)!}{(-1)^n 2^n (x+3)^n / (n)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{2(x+3)}{n+1} \right| = 0$$

The limit is less than 1 for all x-values, so the interval of convergence is  $(-\infty, \infty)$ .

c. Use the Ratio Test to find the interval of convergence.

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-4)^{n+1} / 3^{n+1}}{(n)(x-4)^n / 3^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-4)}{3n} \right| = \frac{|x-4|}{3} \text{ and } \frac{|x-4|}{3} < 1 \text{ when } |x-4| < 3 \text{ or } -3 < x-4 < 3$$

so the interval of convergence includes  $1 < x < 7$ . Check endpoints: when  $x=1$  we have  $\sum_{n=1}^{\infty} (-1)^n n$  and

when  $x=7$  we have  $\sum_{n=1}^{\infty} n$ ; both diverge (by the divergence test), so IoC is  $(1, 7)$ .

d. Use the Ratio Test to find the interval of convergence.

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 3^{n+1} (x+5)^{n+1}}{(-1)^n 3^n (x+5)^n} \right| = \lim_{n \rightarrow \infty} |3(x+5)| < 1. \text{ Now, } |3(x+5)| < 1 \Rightarrow -5\frac{1}{3} < x < -4\frac{2}{3}. \text{ When } x = -5\frac{1}{3}$$

every term in the series is equal to one so it does not converge. And when  $x = -4\frac{2}{3}$  the series becomes

$\sum (-1)^n$  so it does not converge. Thus the interval of convergence is  $-5\frac{1}{3} < x < -4\frac{2}{3}$ .

$$8. \quad \frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} \quad \text{at } \theta = \frac{\pi}{6}$$

$$\frac{dy}{dx} = \frac{3 \sin(3\theta) \sin \theta + \cos(3\theta) \cos \theta}{3 \sin(3\theta) \cos \theta - \cos(3\theta) \sin \theta} \quad \frac{dy}{dx} = \frac{3 \sin\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{6}\right)}{3 \sin\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{6}\right)}$$

So  $\frac{dy}{dx} = \frac{1}{\sqrt{3}}$ . The tangent line at  $\theta = \frac{\pi}{6}$  passes through  $(0,0)$  so the equation of the tangent line

$$y = \frac{1}{\sqrt{3}}x$$

$$9. \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}, \text{ and therefore } e^{-3x} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \dots + \frac{(-3x)^n}{n!} = \sum \frac{(-1)^n 3^n x^n}{n!}.$$

Using the ratio test to find where the series converges:

$$\lim_{n \rightarrow \infty} \left| \frac{(3x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(3x)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3x}{n+1} \right| = 0 < 1. \text{ Therefore, the series converges for all real numbers.}$$

$$10. \quad \text{Avg temp is } \left( \frac{1}{18-0} \right) \int_0^{18} 24 + 9t e^{-(t/3)} dt = \frac{1}{18} \left[ 24t - 27(t+3)e^{-(t/3)} \right]_0^{18} = \frac{513 - 567/e^6}{18} \approx 28.42^\circ \text{C}$$

11. i. a) is a set of functions whose derivative is  $f(x)$ , b) is a numerical value (the net change in  $F(x)$  from  $x = 1$  to  $x = 3$  where  $F'(x) = f(x)$ , and c) is the unique function whose derivative is  $f(x)$  and that contains the point  $(0,0)$ . You should be able to elaborate on each of these and give more specific information.

ii.  $\lim_{\Delta x \rightarrow 0} \sum_{k=0}^{n-1} f(x_k) \Delta x$  is equal to the integral in part b), if the limit exists. It also must be true that

$$\Delta x = \frac{3-1}{n}, \quad x_0 = 1, \quad x_n = x_{n-1} + \Delta x \text{ and } x_n = 3.$$

12. (a) With slope 0.6 and intercept 0, the differential equation must be  $\frac{dD}{dt} = 0.06t$ .

$$\int dD = \int 0.6t dx$$

$D = 0.3t^2 + C$ . Since  $D = 0$  when  $t = 0$ , the constant  $C$  must equal zero, leading to  $D = 0.6t^2$ . Note that you must include the integration constant  $C$  to receive full credit on a test!

(b) With slope 0.05 and intercept 100, the differential equation must be  $\frac{dP}{dt} = 0.05P + 100$ . 0.05 means that

the "natural" growth rate is 5% per year; 100 means that the immigration rate is a constant 100 people per year. Separating variables gives

$$\int \frac{1}{0.05P+100} dP = \int dt$$

$$\frac{1}{0.05} \ln|0.05P+100| = t + C$$

$$\ln|0.05P+100| = 0.05t + C$$

$$|0.05P+100| = Ce^{0.05t}$$

$$P = Ce^{0.05t} - 2000$$

The initial condition given is  $P = 1200$  at time  $t = 0$ , which tells us that this last  $C = 3200$ , giving

$$P = 3200e^{0.05t} - 2000.$$

13.  $F'(x) = \sec^5(x^3) \cdot 3x^2$

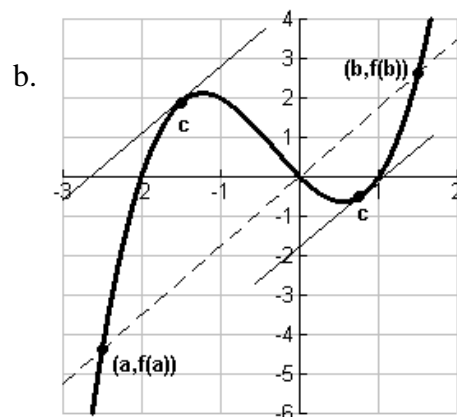
14. a. If a function  $f(x)$  is continuous on  $[a,b]$  and differentiable on  $(a,b)$  there exists a value  $c$  in that interval such

$$\text{that } f'(c) = \frac{f(b) - f(a)}{b - a}.$$

c.  $f(2) = 2.5$  and  $f(4) = 4.25$ , so  $\frac{f(b) - f(a)}{b - a} = 7/8$ .

Differentiating gives  $f'(x) = 1 - \frac{1}{x^2}$ ; to get the desired  $c$ ,

solve  $f'(c) = 7/8$ . Doing so we see that  $c = 2\sqrt{2}$ . Observe that  $2 < 2\sqrt{2} < 4$ , as was claimed by the Mean Value Theorem.



15. volume =  $\int_0^4 \frac{1}{2} (\sqrt{4-x})^2 dx = 4$

16. a.  $\int_0^{24} (150t^2 + 100t + 5000) dt = 840,000$

b.  $2 \cdot 840,000 = \int_0^k (150t^2 + 100t + 5000) dt; k \approx 30.9 \text{ hours}$

c.  $W(t) = \int_0^t (150x^2 + 100x + 5000) dx = 50x^3 + 50x^2 + 5000x \Big|_0^t$   
 $= 50t^3 + 50t^2 + 5000t$

17.  $\frac{1}{2} \int_{-\pi/3}^{\pi/3} [(3 + 2 \cos \theta)^2 - 2^2] d\theta \approx 18.59$

18. a.  $G(0) = \int_{-2}^0 f(t) dt = \frac{1}{4}(\pi \cdot 2^2) = \pi$ . The area from -2 to 0 on the graph is one-quarter of the circle with center at (-2,0) and radius 2.

b.  $G'(1) = -1$ . Since  $G'(x) = f(x)$ , then  $G'(1) = f(1)$ .

c.

