Multiple Integrals and Vector Fields Exam Review Solutions **May 2008**

1a) Find $\lim_{(x,y)\to(0,0)} f(x,y)$

Consider the path $y=0$, $\lim_{(x,0)\to(0,0)} \frac{x^2 y}{x^3+y^3}=0$

path $y=x$, $\lim_{(x,x)\to(0,0)} \frac{x^3}{2x^3}=\frac{1}{2}$

Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

1b) $f(x,y)$ is not continuous $(0,0)$ since $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

1c) Find $\frac{\partial f}{\partial x}(0,0)$.

Using the limit definition of derivative

$$\frac{\partial f}{\partial x}(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}$$

$$\lim_{\Delta x\to 0}\frac{\frac{(0+\Delta x)^2(0)}{(0+\Delta x)^3+(0)^3}-0}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{0}{\Delta x^3}}{\Delta x}=\lim_{\Delta x\to 0}\frac{0}{\Delta x^4}=0$$

2) Show that $f(x+\Delta x, y+\Delta y)-f(x,y)$ can be written as $f_x(x,y)\Delta x+f_y(x,y)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$ where $\lim_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_1=\lim_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_2=0$.

$$\Delta z=f(x+\Delta x, y+\Delta y)-f(x,y)$$

$$\Delta z=-(x+\Delta x)^2(y+\Delta y)+2(y+\Delta y)+x^2y-2y-\left(x^2+2x\Delta x+(\Delta x)^2\right)(y+\Delta y)$$

$$+2y+2\Delta y+x^2y-2y-x^2y-x^2(\Delta y)-2xy(\Delta x)-2x(\Delta x)(\Delta y)-y(\Delta x)^2-(\Delta x)^2\Delta y+2\Delta y+x^2y$$

now with simplification and $f_x=-2xy$ and $f_y=-x^2+2$

$$\Delta z=(-2xy)\Delta x+\left(-x^2+2\right)\Delta y-(2x\Delta y+y\Delta x)\Delta x-(\Delta x)^2\Delta y$$

$$\Delta z=f_x\Delta x+f_y\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y \qquad \text{(Answers may vary!!)}$$

where $\varepsilon_1=-2x\Delta y+y\Delta x$, $\varepsilon_2=-\Delta x^2$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_1=\lim_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_2=0$$

Therefore $f(x,y)$ is differentiable for all (x,y).

3a) Prove $\lim_{x\to -1}(5x+2)=-3$

Show that given any $\varepsilon>0$, there exists a $\delta>0$ such that if $|x-(-1)|<\delta$ then $|5x+2-(-3)|<\varepsilon$. Consider $|5x+2-(-3)|=|5x+5|=5|x+1|$

$5|x+1|<5\delta,$ since $|x+1|<\delta$

now given any $\varepsilon>0,$ choose $\delta=\dfrac{3}{5}$ so that if $|x+1|<\delta,$ then

$|5x+2-(-3)|=5|x+1|<5\delta=5\left(\dfrac{\varepsilon}{5}\right)=\varepsilon$

3b) Prove $\lim_{x\to 2}\left(\dfrac{1}{x}\right)=\dfrac{1}{2}$. Show that given any $\varepsilon>0,$ there exists a $\delta>0$ such that if $|x-2|<\delta$ then $\left|\dfrac{1}{x}-\dfrac{1}{2}\right|<\varepsilon.$

Consider $\left|\dfrac{1}{x}-\dfrac{1}{2}\right|=\left|\dfrac{2-x}{2x}\right|<\dfrac{1}{2}\left|\dfrac{1}{x}\right|\delta$ since $|x-2|<\delta.$

Consider x's "close to" 2. $1\le \text{x} \le 3$ then δ=1 then $1\ge \dfrac{1}{\text{x}}\ge\dfrac{1}{3}$ so $\left|\dfrac{1}{\text{x}}\right|\le 1$

So from above we have $\left|\dfrac{1}{x}-\dfrac{1}{2}\right|<\dfrac{1}{2}\left|\dfrac{1}{x}\right|\delta\le\dfrac{1}{2}\delta$

Given $\varepsilon>0$, choose $\delta=\min\{1,2\varepsilon\}$. Then if $|x-2|<\delta,$

we have $\left|\dfrac{1}{x}-\dfrac{1}{2}\right|<\dfrac{1}{2}\left|\dfrac{1}{x}\right|\delta\le\dfrac{1}{2}\delta=\dfrac{1}{2}(2\varepsilon)=\varepsilon$

4a) $V=\int_1^3\int_0^2(3x^3+3x^2y)dy\,dx=\int_1^3(6x^3+6x^2)dx=172$

4b) $V=\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(3-x)dy\,dx=\int_0^{2\pi}\int_0^3(3r-r^2\cos\theta)drd\theta=\int_0^{2\pi}\dfrac{27}{2}-9\cos\theta d\theta=27\pi$

5a) $\int_0^{\frac{\pi}{4}}\int_0^2\dfrac{r}{1+r^2}dr\,d\theta=\dfrac{1}{2}\ln 5\int_0^{\frac{\pi}{4}}d\theta=\dfrac{\pi}{8}\ln 5$

5b) $\int_0^{\frac{\pi}{2}}\int_0^1\cos(r^2)r\,dr\,d\theta=\dfrac{1}{2}\sin 1\int_0^{\frac{\pi}{2}}d\theta=\dfrac{\pi}{4}\sin 1$

5c) $\int_0^1\int_0^{2x}\cos(x^2)dy\,dx=\int_0^1 2x\cos(x^2)dx=\sin 1$

5d) $\int_0^2 \int_0^{x^2} e^{x^3} dy\, dx = \int_0^2 x^2 e^{x^3} dx = \frac{\left(e^8 - 1\right)}{3}$

5e) $\int_0^{\sqrt{2}} \int_0^x \int_0^{2-x^2} xyz\, dz\, dy\, dx = \int_0^{\sqrt{2}} \int_0^x \frac{1}{2} xy\left(2 - x^2\right)^2 dy\, dx = \int_0^{\sqrt{2}} \frac{1}{4} x^3 \left(2 - x^2\right)^2 dx = \frac{1}{6}$

5f) $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \sin\phi \cos\phi\, d\rho\, d\phi\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{4} \sin\phi \cos\phi\, d\phi\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{8} d\theta = \frac{\pi}{16}$

5g) $\int_0^{\frac{\pi}{2}} \int_0^a \int_0^{a^2 - r^2} r^3 \cos^2\theta\, dz\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^a \left(a^2 r^3 - r^5\right) \cos^2\theta\, dr\, d\theta = \frac{1}{12} a^6 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \frac{\pi a^6}{48}$

5h) $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{8}} \rho^4 \cos^2\phi \sin\phi\, d\rho\, d\phi\, d\theta = \frac{32\left(2\sqrt{2} - 1\right)\pi}{15}$

6a) $C: x = t^2, y = t, 0 \le t \le 1;\ W = \int_0^1 3t^4 dt = \frac{3}{5}$

6b) $W = \int_0^1 \left(t^3 + 5t^6\right) dt = \frac{27}{28}$

7) By Stokes' Theorem,

$$\oiint_S curl\, \vec{F}\, dS = \iint_R \left(1 + 2y\right) dA = \int_0^{2\pi} \int_0^1 \left(r + 2r^2 \sin\theta\right) dr\, d\theta = \pi$$

8) By Divergence Theorem,

$$\iiint_V \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} dV = \iiint_V \left(-2xz - z + 2z\right) dV = \iiint_V 0\, dV = 0$$

9) $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, so $\int_0^1 \left(2t - 3t^2\right)\sqrt{4 + 36t^2}\, dt = \frac{-11}{108}\sqrt{10} - \frac{1}{36}\ln\left(\sqrt{10} - 3\right) - \frac{4}{27}$

10a) $C: x = t,\ y = t,\ 0 \le t \le 1;\ \int_0^1 6t\, dt = 3$

10b) $C: x = t,\ y = t,\ 0 \le t \le 1;$

$$\int_0^1 \left[3t + 2\sin\left(\frac{\pi t}{2}\right) + \pi t \cos\left(\frac{\pi t}{2}\right) - \left(\frac{\pi}{2}\right)\sin\left(\frac{\pi t}{2}\right)\cos\left(\frac{\pi t}{2}\right)\right] dt = 3$$

11) $\int_0^{\pi} F \bullet dr = \int_0^{\pi} < 4\cos^2 t, 4\cos t \sin t > \bullet < -2\sin t, 2\cos t > = \int_0^{\pi} (0)\, dt = 0$

12) $$\frac{\partial(\cos y + y\cos x)}{\partial y} = -\sin y + \cos x = \frac{\partial(\sin x - x\sin y)}{\partial x}, \quad \text{conservative so}$$

$$\frac{\partial\theta}{\partial x} = \cos y + y\cos x \text{ and } \frac{\partial\phi}{\partial y} = \sin x - x\sin y,\ \phi = x\cos y + y\sin x + k(x)$$

$$\phi = x\cos y + y\sin x + K(x)$$

$$\theta = x\cos y + y\sin x + g(y)$$

$$\textit{Potential Function},\ f(x,y) = x\cos y + y\sin x + C$$

13 a)

$$\frac{\partial(e^x \sin y)}{\partial y} = e^x \cos y = \frac{\partial(e^x \cos y)}{\partial x}, \textit{By the Fundamental Theorem of Line Integrals}:$$

$$\phi = e^x \sin y,\ \phi\left(1, \frac{\pi}{2}\right) - \phi(0,0) = e$$

13 b)

$$\frac{\partial(2xy^3)}{\partial y} = 6xy^2 = \frac{\partial(3x^2y^2)}{\partial x} \textit{ so the field is conservative.}$$

$$\textit{So use the Fundamental Theorem of Line Integrals}:$$

$$f(x,y) = x^2y^3,\ f(-1,0) - f(2,-2) = 32$$

14) Show the field is conservative (and so is independent of path), then use Fund Thm of Line Integrals:

$$f(x,y) = e^{xy} + c$$

$$f(2,0) - f(-1,1) = e^0 - e^{-1} = 1 - e^{-1} = \frac{e-1}{e}$$

15a) By Green's Theorem $\int_0^1 \int_{x^2}^{x} (2x - 2y)\, dy\, dx = \frac{1}{30}$

15b) By Green's Theorem $\iint_R (\cos x \cos y - \cos x \cos y)\, dA = 0$

16)

$$\frac{\sqrt{29}}{16} \int_0^6 \int_0^{\frac{(12-2x)}{3}} xy(12 - 2x - 3y)\, dy\, dx = \frac{\sqrt{29}}{16} \int_0^6 \int_0^{\frac{(12-2x)}{3}} 12xy - 2x^2y - 3xy^2 dy\, dx$$

$$= \frac{\sqrt{29}}{16} \int_0^6 6xy^2 - x^2y^2 - xy^3 \Big|_0^{12-2x} dx =$$

$$= \frac{\sqrt{29}}{16} \int_0^6 864x - 432x^2 + 24x^3 + 48x^3 - 4x^4 - x(144 - 48x^2 + 4x^2)(12 - 2x)\, dx$$

$$= \frac{\sqrt{29}}{16} \int_0^6 -864x + 432x^2 - 72x^3 + 4x^4)\, dx = \frac{\sqrt{29}}{16}(\text{-1 555.2}) \cong 523.4$$

17a) $\mathbf{n} = -z_x\mathbf{i} - z_y\mathbf{j} + \mathbf{k}$, $\iint_R \mathbf{F}\bullet\mathbf{n}\, dS = \iint_R \left(2x^2 + 2y^2 + \left(1 - x^2 - y^2\right)\right) dS = \int_0^{2\pi}\int_0^1 2r\, dr\, d\theta = 2\pi$

17b) R is the annular region enclosed by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

$$\iint_\sigma \mathbf{F}\bullet\mathbf{n}\, dS = \iint_R \left(-\frac{x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} + 2z\right) dA = \iint_R \sqrt{x^2+y^2}\, dA = \int_0^{2\pi}\int_1^2 r^2 dr\, d\theta = \frac{14\pi}{3}$$

18a) G is the rectangular solid; $\iiint_G div\,\mathbf{F}\, dV = \int_0^2\int_0^1\int_0^3 (2x-1)\, dx\, dy\, dz = 12$

18b) G is the cylindrical solid;

$$\iiint_G div\,\mathbf{F}\, dV = 3\iiint_G dV = (3)(\text{volume of cylinder}) = 3\left[\pi a^2(1)\right] = 3\pi a^2$$

18c) G is the solid bounded by $z = 1 - x^2 - y^2$ and the *xy*-plane;

$$\iiint_G div\,\mathbf{F}\, dV = 3\iiint_G dV = 3\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\, dz\, dr\, d\theta = \frac{3\pi}{2}$$

19) If σ is oriented with upward normals then C consists of three parts parameterized as

$C_1 : r(t) = (1-t)\mathbf{i} + t\mathbf{j}$ for $0 \le t \le 1$,

$C_2 : r(t) = (1-t)\mathbf{i} + t\mathbf{k}$ for $0 \le t \le 1$,

$C_3 : r(t) = t\mathbf{i} + (1-t)\mathbf{k}$ for $0 \le t \le 1$.

$$\int_{C_1} \mathbf{F}\bullet d\mathbf{r} = \int_{C_2} \mathbf{F}\bullet d\mathbf{r} = \int_{C_3} \mathbf{F}\bullet d\mathbf{r} = \int_1^0 (3t-1)dt = -\frac{1}{2} \text{ or } -\int_0^1 (3t-1)dt = -\frac{1}{2}$$

so $\oint_C \mathbf{F}\bullet d\mathbf{r} = -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{3}{2}$. $curl\,\mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, $z = 1 - x - y$,

R is the triangular region in the *xy*-plane enclosed by $x + y = 1$, $x = 0$, and $y = 0$;

$$\iint_\sigma (curl\,\mathbf{F})\bullet\mathbf{n}\, dS = -3\iint_R dA = (-3)(\text{area of R}) = (-3)\left[\frac{1}{2}(1)(1)\right] = \frac{-3}{2}$$

20) $curl\,\mathbf{F} = -4\mathbf{i} - 6\mathbf{j} + 6y\mathbf{k}$, $z = \frac{y}{2}$ oriented with upward normals,

R is the triangular region in the *xy*-plane enclosed by $x + y = 2$, $x = 0$, and $y = 0$;

$$\iint_\sigma (curl\,\mathbf{F})\bullet\mathbf{n}\, dS = \iint_R (3 + 6y)\, dA = \int_0^2\int_0^{2-x} (3 + 6y)\, dy\, dx = 14$$