Calculus Challenge #1
Solution

Solution due: October 21, 2009

In your calculus class you have used the definition of derivative to develop the rules for differentiating the basic functions from precalculus. You used the definition to prove that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$ by multiplying the difference quotient $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ by a clever form of 1 and simplifying before taking the limit.

Modify that technique to find the derivatives of the following functions *using the definition of derivative*.

> This solution contains a lot of detail. Students are not expected to write all this. Your solution does not need to mimic ours to be correct. Your solution may well be better! We want the solutions to have enough detail so, if students were stuck on a problem, they can see how to proceed. Consequently, our posted solutions are likely to be much longer than yours.

a) $\frac{d}{dx}\sqrt[3]{x}$

When we multiplied $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ by $\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$ we were using the difference of square factoring procedure $(a-b)(a+b)=a^2-b^2$ to simplify the numerator. The principal goal of this rewriting of the difference quotient is to eliminate the *h* in the denominator. Once that is done, we can take the limit as *h* goes to zero without a problem.

So, we modify this method for the difference of two cubes. By division, we see that

$$\frac{a^3-b^3}{(a-b)} = a^2+ab+b^2, \text{ so } (a-b)\left(a^2+ab+b^2\right) = a^3-b^3.$$

$\frac{d}{dx}\sqrt[3]{x} = \lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$. To make our work less cluttered and more easily read, we will simplify the difference quotient, then take the limit of the result.

$$\left(\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}\right)\cdot\left(\frac{\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2}{\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2}\right)=$$

$$\left(\frac{\left(\sqrt[3]{x+h}\right)^3+\left(\sqrt[3]{x+h}\right)^2\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)^2-\left(\sqrt[3]{x+h}\right)^2\left(\sqrt[3]{x}\right)-\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)^2-\left(\sqrt[3]{x}\right)^3}{(h)\ \left(\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2\right)}\right)$$

We see that the terms in the numerator add out as expected and the remaining terms are cubes of cube roots,

$$\left(\frac{\left(\sqrt[3]{x+h}\right)^3+\left(\sqrt[3]{x+h}\right)^2\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)^2-\left(\sqrt[3]{x+h}\right)^2\left(\sqrt[3]{x}\right)-\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)^2-\left(\sqrt[3]{x}\right)^3}{h\ \left(\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2\right)}\right)$$

leaving just

$$\left(\frac{(x+h)-(h)}{h\ \left(\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2\right)}\right)=\left(\frac{h}{h\ \left(\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2\right)}\right)$$

and we see the common factor of *h* in the numerator and denominator.

So, we have

$$\lim_{h\to 0}\left(\frac{1}{\left(\sqrt[3]{x+h}\right)^2+\left(\sqrt[3]{x+h}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2}\right)=\frac{1}{\left(\sqrt[3]{x}\right)^2+\left(\sqrt[3]{x}\right)\left(\sqrt[3]{x}\right)+\left(\sqrt[3]{x}\right)^2}=\frac{1}{3\left(\sqrt[3]{x}\right)^2}.$$

We have proven that $\frac{d}{dx}\sqrt[3]{x}=\frac{1}{3\left(\sqrt[3]{x}\right)^2}$ using the definition of derivative.

b) $\frac{d}{dx}\sqrt[n]{x}$ for positive integer values of *n*.

Before trying the n^{th} root, we should look at the 4^{th} and 5^{th} roots. By division, we see that $\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3$ and $\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4$. Notice, as before, the sum of the exponents in each term is one less than the power and all coefficients are positive one. The expressions that follow would be much shorter if we used summation notation, but it is a bit easier to read and see the patterns in expanded form. We have also written out more terms than necessary but we want to make the pattern obvious.

Both

$$\left(\frac{\sqrt[4]{x+h}-\sqrt[4]{x}}{h}\right)\cdot\left(\frac{\left(\sqrt[4]{x+h}\right)^3+\left(\sqrt[4]{x+h}\right)^2\left(\sqrt[4]{x}\right)+\left(\sqrt[4]{x+h}\right)\left(\sqrt[4]{x}\right)^2+\left(\sqrt[4]{x}\right)^3}{\left(\sqrt[4]{x+h}\right)^3+\left(\sqrt[4]{x+h}\right)^2\left(\sqrt[4]{x}\right)+\left(\sqrt[4]{x+h}\right)\left(\sqrt[4]{x}\right)^2+\left(\sqrt[4]{x}\right)^3}\right)$$

and

$$\left(\frac{\sqrt[5]{x+h}-\sqrt[5]{x}}{h}\right)\cdot\left(\frac{\left(\sqrt[5]{x+h}\right)^4+\left(\sqrt[5]{x+h}\right)^3\left(\sqrt[5]{x}\right)+\left(\sqrt[5]{x+h}\right)^2\left(\sqrt[5]{x}\right)^2+\left(\sqrt[5]{x+h}\right)\left(\sqrt[5]{x}\right)^3+\left(\sqrt[5]{x}\right)^4}{\left(\sqrt[5]{x+h}\right)^4+\left(\sqrt[5]{x+h}\right)^3\left(\sqrt[5]{x}\right)+\left(\sqrt[5]{x+h}\right)^2\left(\sqrt[5]{x}\right)^2+\left(\sqrt[5]{x+h}\right)\left(\sqrt[5]{x}\right)^3+\left(\sqrt[5]{x}\right)^4}\right)$$

work out as expected and each gives the correct result.

So, $\frac{d}{dx}\sqrt[n]{x}=\lim_{h\to 0}\frac{\sqrt[n]{x+h}-\sqrt[n]{x}}{h}$ can be evaluated by rationalizing the numerator, dividing out the resulting common factor of *h* in the numerator and denominator, and then taking the limit as *h* goes to zero.

$$\left(\frac{\sqrt[n]{x+h}-\sqrt[n]{x}}{h}\right)\cdot\left(\frac{\left(\sqrt[n]{x+h}\right)^{n-1}+\left(\sqrt[n]{x+h}\right)^{n-2}\left(\sqrt[n]{x}\right)+\left(\sqrt[n]{x+h}\right)^{n-3}\left(\sqrt[n]{x}\right)^2+\cdots+\left(\sqrt[n]{x+h}\right)^2\left(\sqrt[n]{x}\right)^{n-3}+\left(\sqrt[n]{x+h}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}{\left(\sqrt[n]{x+h}\right)^{n-1}+\left(\sqrt[n]{x+h}\right)^{n-2}\left(\sqrt[n]{x}\right)+\left(\sqrt[n]{x+h}\right)^{n-3}\left(\sqrt[n]{x}\right)^2+\cdots+\left(\sqrt[n]{x+h}\right)^2\left(\sqrt[n]{x}\right)^{n-3}+\left(\sqrt[n]{x+h}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}\right)$$

When we multiply the numerators, all terms except the first and last add out, and the first and last are $\left(\sqrt[n]{x+h}\right)^n$ and $-\left(\sqrt[n]{x}\right)^n$. To see this, consider the general terms $\cdots\left(\sqrt[n]{x+h}\right)^k\left(\sqrt[n]{x}\right)^{n-k-1}+\left(\sqrt[n]{x+h}\right)^{k-1}\left(\sqrt[n]{x}\right)^{n-k}+\cdots$ are both multiplied by $\left(\sqrt[n]{x+h}\right)$ and $-\left(\sqrt[n]{x}\right)$. The right term $\left(\sqrt[n]{x+h}\right)^{k-1}\left(\sqrt[n]{x}\right)^{n-k}$ when multiplied by $\left(\sqrt[n]{x+h}\right)$ is $\left(\sqrt[n]{x+h}\right)^k\left(\sqrt[n]{x}\right)^{n-k}$, the same value but opposite sign as the left term $\left(\sqrt[n]{x+h}\right)^k\left(\sqrt[n]{x}\right)^{n-k-1}$ when multiplied by $-\left(\sqrt[n]{x}\right)$. This happens for all interior products.

So,

$$\left(\frac{\sqrt[n]{x+h}-\sqrt[n]{x}}{h}\right)\cdot\left(\frac{\left(\sqrt[n]{x+h}\right)^{n-1}+\left(\sqrt[n]{x+h}\right)^{n-2}\left(\sqrt[n]{x}\right)+\left(\sqrt[n]{x+h}\right)^{n-3}\left(\sqrt[n]{x}\right)^{2}+\cdots+\left(\sqrt[n]{x+h}\right)^{2}\left(\sqrt[n]{x}\right)^{n-3}+\left(\sqrt[n]{x+h}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}{\left(\sqrt[n]{x+h}\right)^{n-1}+\left(\sqrt[n]{x+h}\right)^{n-2}\left(\sqrt[n]{x}\right)+\left(\sqrt[n]{x+h}\right)^{n-3}\left(\sqrt[n]{x}\right)^{2}+\cdots+\left(\sqrt[n]{x+h}\right)^{2}\left(\sqrt[n]{x}\right)^{n-3}+\left(\sqrt[n]{x+h}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}\right)$$

reduces to

$$\left(\frac{1}{h}\right)\cdot\left(\frac{\left(\sqrt[n]{x+h}\right)^{n}-\left(\sqrt[n]{x}\right)^{n}}{\left(\sqrt[n]{x+h}\right)^{n-1}+\left(\sqrt[n]{x+h}\right)^{n-2}\left(\sqrt[n]{x}\right)+\left(\sqrt[n]{x+h}\right)^{n-3}\left(\sqrt[n]{x}\right)^{2}+\cdots+\left(\sqrt[n]{x+h}\right)^{2}\left(\sqrt[n]{x}\right)^{n-3}+\left(\sqrt[n]{x+h}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}\right)$$

and to

$$\left(\frac{1}{\left(\sqrt[n]{x+h}\right)^{n-1}+\left(\sqrt[n]{x+h}\right)^{n-2}\left(\sqrt[n]{x}\right)+\left(\sqrt[n]{x+h}\right)^{n-3}\left(\sqrt[n]{x}\right)^{2}+\cdots+\left(\sqrt[n]{x+h}\right)^{2}\left(\sqrt[n]{x}\right)^{n-3}+\left(\sqrt[n]{x+h}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}\right)$$

Now, taking the limit, we have

$$\lim_{h\to 0}\left(\frac{1}{\left(\sqrt[n]{x+h}\right)^{n-1}+\left(\sqrt[n]{x+h}\right)^{n-2}\left(\sqrt[n]{x}\right)+\left(\sqrt[n]{x+h}\right)^{n-3}\left(\sqrt[n]{x}\right)^{2}+\cdots+\left(\sqrt[n]{x+h}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}\right)=$$

$$\left(\frac{1}{\left(\sqrt[n]{x}\right)^{n-1}+\left(\sqrt[n]{x}\right)^{n-2}\left(\sqrt[n]{x}\right)+\left(\sqrt[n]{x}\right)^{n-3}\left(\sqrt[n]{x}\right)^{2}+\cdots+\left(\sqrt[n]{x}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}\right)=\frac{1}{n\left(\sqrt[n]{x}\right)^{n-1}}.$$

c) Would the process used in b) work if n was a negative integer? Explain why or why not.

Given $\sqrt[n]{x}$, if n is negative, we have $\frac{d}{dx}\frac{1}{\sqrt[n]{x}}=\lim\limits_{h\to 0}\frac{\left(\frac{1}{\sqrt[n]{x+h}}-\frac{1}{\sqrt[n]{x}}\right)}{h}$. The first step would be to rewrite the numerator as a single rational expression by performing the indicated subtraction.

So, $\frac{\left(\frac{1}{\sqrt[n]{x+h}}-\frac{1}{\sqrt[n]{x}}\right)}{h}=\left(\frac{1}{h}\right)\left(\frac{\sqrt[n]{x}-\sqrt[n]{x+h}}{\sqrt[n]{x+h}\cdot\sqrt[n]{x}}\right)$. Now, we don't really worry about the denominator.

We can simplify the numerator as before. This new numerator is just -1 times the old one, so the result will be the opposite of what we saw in part b).

$$\left(\frac{1}{h}\right)\left(\frac{\sqrt[n]{x}-\sqrt[n]{x+h}}{\sqrt[n]{x+h}\cdot\sqrt[n]{x}}\right)\cdot\left(\frac{\left(\sqrt[n]{x+h}\right)^{n-1}+\left(\sqrt[n]{x+h}\right)^{n-2}\left(\sqrt[n]{x}\right)+\cdots+\left(\sqrt[n]{x+h}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}{\left(\sqrt[n]{x+h}\right)^{n-1}+\left(\sqrt[n]{x+h}\right)^{n-2}\left(\sqrt[n]{x}\right)+\cdots+\left(\sqrt[n]{x+h}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}\right)$$ simplifies to

$$\left(\frac{1}{h}\right)\left(\frac{1}{\sqrt[n]{x+h}\cdot\sqrt[n]{x}}\right)\cdot\left(\frac{-h}{\left(\sqrt[n]{x+h}\right)^{n-1}+\left(\sqrt[n]{x+h}\right)^{n-2}\left(\sqrt[n]{x}\right)+\cdots+\left(\sqrt[n]{x+h}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}\right)$$

Dividing out the *h*, and taking the limit as *h* approaches zero, gives

$$\left(\frac{1}{\sqrt[n]{x}\cdot\sqrt[n]{x}}\right)\cdot\left(\frac{-1}{\left(\sqrt[n]{x}\right)^{n-1}+\left(\sqrt[n]{x}\right)^{n-2}\left(\sqrt[n]{x}\right)+\cdots+\left(\sqrt[n]{x}\right)\left(\sqrt[n]{x}\right)^{n-2}+\left(\sqrt[n]{x}\right)^{n-1}}\right)=\frac{-1}{\left(\sqrt[n]{x}\right)^{2}\left(n\left(\sqrt[n]{x}\right)^{n-1}\right)}=\frac{-1}{n\left(\sqrt[n]{x}\right)^{n+1}}.$$

This may be more commonly written as $\frac{d}{dx}\frac{1}{\sqrt[n]{x}}=\frac{-1}{n\left(\sqrt[n]{x}\right)^{n+1}}=-\left(\frac{1}{n}\right)x^{\left(\frac{-n-1}{n}\right)}$.

d) $\frac{d}{dx}\sqrt[5]{x^3}$ Now, this is much more challenging. By playing around, and paying attention to why and in what way our first attempts didn't work, we see that we need to rewrite $\left(a^{\frac{3}{5}}-b^{\frac{3}{5}}\right)\left(a^{\frac{12}{5}}+a^{\frac{9}{5}}b^{\frac{3}{5}}+a^{\frac{6}{5}}b^{\frac{6}{5}}+a^{\frac{3}{5}}b^{\frac{9}{5}}+b^{\frac{12}{5}}\right)=a^{\frac{15}{5}}-b^{\frac{15}{5}}$ which should simplify. So, in our case we have $a=x+h$ and $b=x$. Simplifying the difference quotient, we find

$$\left(\frac{\sqrt[5]{(x+h)^3}-\sqrt[5]{x^3}}{h}\right)\cdot\left(\frac{\left(\sqrt[5]{(x+h)^3}\right)^4+\left(\sqrt[5]{(x+h)^3}\right)^3\left(\sqrt[5]{x^3}\right)+\left(\sqrt[5]{(x+h)^3}\right)^2\left(\sqrt[5]{x^3}\right)^2+\left(\sqrt[5]{(x+h)^3}\right)\left(\sqrt[5]{x^3}\right)^3+\left(\sqrt[5]{x^3}\right)^4}{\left(\sqrt[5]{(x+h)^3}\right)^4+\left(\sqrt[5]{(x+h)^3}\right)^3\left(\sqrt[5]{x^3}\right)+\left(\sqrt[5]{(x+h)^3}\right)^2\left(\sqrt[5]{x^3}\right)^2+\left(\sqrt[5]{(x+h)^3}\right)\left(\sqrt[5]{x^3}\right)^3+\left(\sqrt[5]{x^3}\right)^4}\right)=$$

$$\left(\frac{1}{h}\right)\cdot\left(\frac{\left(\sqrt[5]{(x+h)^3}\right)^5-\left(\sqrt[5]{x^3}\right)^5}{\left(\sqrt[5]{(x+h)^3}\right)^4+\left(\sqrt[5]{(x+h)^3}\right)^3\left(\sqrt[5]{x^3}\right)+\left(\sqrt[5]{(x+h)^3}\right)^2\left(\sqrt[5]{x^3}\right)^2+\left(\sqrt[5]{(x+h)^3}\right)\left(\sqrt[5]{x^3}\right)^3+\left(\sqrt[5]{x^3}\right)^4}\right)=$$

$$\left(\frac{1}{h}\right)\cdot\left(\frac{(x+h)^3-x^3}{\left(\sqrt[5]{(x+h)^3}\right)^4+\left(\sqrt[5]{(x+h)^3}\right)^3\left(\sqrt[5]{x^3}\right)+\left(\sqrt[5]{(x+h)^3}\right)^2\left(\sqrt[5]{x^3}\right)^2+\left(\sqrt[5]{(x+h)^3}\right)\left(\sqrt[5]{x^3}\right)^3+\left(\sqrt[5]{x^3}\right)^4}\right)=$$

$$\left(\frac{1}{h}\right)\cdot\left(\frac{\left(x^3+3x^2h+3xh^2+h^3-x^3\right)}{\left(\sqrt[5]{(x+h)^3}\right)^4+\left(\sqrt[5]{(x+h)^3}\right)^3\left(\sqrt[5]{x^3}\right)+\left(\sqrt[5]{(x+h)^3}\right)^2\left(\sqrt[5]{x^3}\right)^2+\left(\sqrt[5]{(x+h)^3}\right)\left(\sqrt[5]{x^3}\right)^3+\left(\sqrt[5]{x^3}\right)^4}\right)=$$

$$\left(\frac{\left(3x^2+3xh+h^2\right)}{\left(\sqrt[5]{(x+h)^3}\right)^4+\left(\sqrt[5]{(x+h)^3}\right)^3\left(\sqrt[5]{x^3}\right)+\left(\sqrt[5]{(x+h)^3}\right)^2\left(\sqrt[5]{x^3}\right)^2+\left(\sqrt[5]{(x+h)^3}\right)\left(\sqrt[5]{x^3}\right)^3+\left(\sqrt[5]{x^3}\right)^4}\right)$$

Finally, we take the limit.

$$\lim_{h\to 0}\left(\frac{\left(3x^2+3xh+h^2\right)}{\left(\sqrt[5]{(x+h)^3}\right)^4+\left(\sqrt[5]{(x+h)^3}\right)^3\left(\sqrt[5]{x^3}\right)+\left(\sqrt[5]{(x+h)^3}\right)^2\left(\sqrt[5]{x^3}\right)^2+\left(\sqrt[5]{(x+h)^3}\right)\left(\sqrt[5]{x^3}\right)^3+\left(\sqrt[5]{x^3}\right)^4}\right)=$$

$$\lim_{h\to 0}\left(\frac{\left(3x^2\right)}{\left(\sqrt[5]{(x)^3}\right)^4+\left(\sqrt[5]{(x)^3}\right)^3\left(\sqrt[5]{x^3}\right)+\left(\sqrt[5]{(x)^3}\right)^2\left(\sqrt[5]{x^3}\right)^2+\left(\sqrt[5]{(x)^3}\right)\left(\sqrt[5]{x^3}\right)^3+\left(\sqrt[5]{x^3}\right)^4}\right)=\frac{3x^2}{5\left(\sqrt[5]{x^3}\right)^4}=\frac{3x^2}{5x^2\left(\sqrt[5]{x^2}\right)}=\frac{3}{5\left(\sqrt[5]{x^2}\right)}.$$

This is the desired result.

Before we try to generalize this more complicated derivative, let's try $\frac{d}{dx}\sqrt[5]{x^2}$ and $\frac{d}{dx}\sqrt[5]{x^4}$.

Using $\left(a^{\frac{2}{5}}-b^{\frac{2}{5}}\right)\left(a^{\frac{8}{5}}+a^{\frac{6}{5}}b^{\frac{2}{5}}+a^{\frac{4}{5}}b^{\frac{4}{5}}+a^{\frac{2}{5}}b^{\frac{6}{5}}+b^{\frac{8}{5}}\right)=a^{\frac{10}{5}}-b^{\frac{10}{5}}$, we should multiply $\left(\frac{\sqrt[5]{(x+h)^2}-\sqrt[5]{x^2}}{h}\right)$

by $\left(\frac{\left(\sqrt[5]{(x+h)^2}\right)^4+\left(\sqrt[5]{(x+h)^2}\right)^3\left(\sqrt[5]{x^2}\right)+\left(\sqrt[5]{(x+h)^2}\right)^2\left(\sqrt[5]{x^2}\right)^2+\left(\sqrt[5]{(x+h)^2}\right)\left(\sqrt[5]{x^2}\right)^3+\left(\sqrt[5]{x^2}\right)^4}{\left(\sqrt[5]{(x+h)^2}\right)^4+\left(\sqrt[5]{(x+h)^2}\right)^3\left(\sqrt[5]{x^2}\right)+\left(\sqrt[5]{(x+h)^2}\right)^2\left(\sqrt[5]{x^2}\right)^2+\left(\sqrt[5]{(x+h)^2}\right)\left(\sqrt[5]{x^2}\right)^3+\left(\sqrt[5]{x^2}\right)^4}\right)$

Using $\left(a^{\frac{4}{5}}-b^{\frac{4}{5}}\right)\left(a^{\frac{16}{5}}+a^{\frac{12}{5}}b^{\frac{4}{5}}+a^{\frac{8}{5}}b^{\frac{8}{5}}+a^{\frac{4}{5}}b^{\frac{12}{5}}+b^{\frac{16}{5}}\right)=a^{\frac{20}{5}}-b^{\frac{20}{5}}$, we should multiply $\left(\frac{\sqrt[5]{(x+h)^4}-\sqrt[5]{x^4}}{h}\right)$

by $\left(\frac{\left(\sqrt[5]{(x+h)^4}\right)^4+\left(\sqrt[5]{(x+h)^4}\right)^3\left(\sqrt[5]{x^4}\right)+\left(\sqrt[5]{(x+h)^4}\right)^2\left(\sqrt[5]{x^4}\right)^2+\left(\sqrt[5]{(x+h)^4}\right)\left(\sqrt[5]{x^4}\right)^3+\left(\sqrt[5]{x^4}\right)^4}{\left(\sqrt[5]{(x+h)^4}\right)^4+\left(\sqrt[5]{(x+h)^4}\right)^3\left(\sqrt[5]{x^4}\right)+\left(\sqrt[5]{(x+h)^4}\right)^2\left(\sqrt[5]{x^4}\right)^2+\left(\sqrt[5]{(x+h)^4}\right)\left(\sqrt[5]{x^4}\right)^3+\left(\sqrt[5]{x^4}\right)^4}\right)$

Both work.

Also try $\sqrt[7]{x^3}$ and $\sqrt[5]{x^7}$ as well as some even roots, for example, $\sqrt[4]{x^3}$ and $\sqrt[6]{x^3}$ (which should be the same as the square root) to check to see if there are parity issues.

For $\sqrt[7]{x^3}$, we need $\left(a^{\frac{3}{7}}-b^{\frac{3}{7}}\right)\left(a^{\frac{18}{7}}+a^{\frac{15}{7}}b^{\frac{3}{7}}+a^{\frac{12}{7}}b^{\frac{6}{7}}+a^{\frac{9}{7}}b^{\frac{9}{7}}+a^{\frac{6}{7}}b^{\frac{12}{7}}+a^{\frac{3}{7}}b^{\frac{15}{7}}+b^{\frac{18}{5}}\right)=a^{\frac{21}{7}}-b^{\frac{21}{7}}$

$$\left(\frac{\sqrt[7]{(x+h)^3}-\sqrt[7]{x^3}}{h}\right)\cdot\left(\frac{\left(\sqrt[7]{(x+h)^3}\right)^6+\left(\sqrt[7]{(x+h)^3}\right)^5\left(\sqrt[7]{x^3}\right)+\left(\sqrt[7]{(x+h)^3}\right)^4\left(\sqrt[7]{x^3}\right)^2+\cdots+\left(\sqrt[7]{(x+h)^3}\right)\left(\sqrt[7]{x^3}\right)^5+\left(\sqrt[7]{x^3}\right)^6}{\left(\sqrt[7]{(x+h)^3}\right)^6+\left(\sqrt[7]{(x+h)^3}\right)^5\left(\sqrt[7]{x^3}\right)+\left(\sqrt[7]{(x+h)^3}\right)^4\left(\sqrt[7]{x^3}\right)^2+\cdots+\left(\sqrt[7]{(x+h)^3}\right)\left(\sqrt[7]{x^3}\right)^5+\left(\sqrt[7]{x^3}\right)^6}\right)=$$

For $\sqrt[5]{x^7}$, we need $\left(a^{\frac{7}{5}}-b^{\frac{7}{5}}\right)\left(a^{\frac{28}{5}}+a^{\frac{21}{5}}b^{\frac{7}{5}}+a^{\frac{14}{5}}b^{\frac{14}{5}}+a^{\frac{7}{5}}b^{\frac{21}{5}}+b^{\frac{28}{5}}\right)=a^{\frac{35}{5}}-b^{\frac{35}{5}}$

$$\left(\frac{\sqrt[5]{(x+h)^7}-\sqrt[5]{x^7}}{h}\right)\cdot\left(\frac{\left(\sqrt[5]{(x+h)^7}\right)^4+\left(\sqrt[5]{(x+h)^7}\right)^3\left(\sqrt[5]{x^7}\right)+\left(\sqrt[5]{(x+h)^7}\right)^2\left(\sqrt[5]{x^7}\right)^2+\left(\sqrt[5]{(x+h)^7}\right)\left(\sqrt[5]{x^7}\right)^3+\left(\sqrt[5]{x^7}\right)^4}{\left(\sqrt[5]{(x+h)^7}\right)^4+\left(\sqrt[5]{(x+h)^7}\right)^3\left(\sqrt[5]{x^7}\right)+\left(\sqrt[5]{(x+h)^7}\right)^2\left(\sqrt[5]{x^7}\right)^2+\left(\sqrt[5]{(x+h)^7}\right)\left(\sqrt[5]{x^7}\right)^3+\left(\sqrt[5]{x^7}\right)^4}\right)$$

For $\sqrt[4]{x^3}$, we need $\left(a^{\frac{3}{4}}-b^{\frac{3}{4}}\right)\left(a^{\frac{9}{4}}+a^{\frac{6}{4}}b^{\frac{3}{4}}+a^{\frac{3}{4}}b^{\frac{6}{4}}+b^{\frac{9}{4}}\right)=a^{\frac{12}{4}}-b^{\frac{12}{4}}$

$$\left(\frac{\sqrt[4]{(x+h)^3}-\sqrt[4]{x^3}}{h}\right)\cdot\left(\frac{\left(\sqrt[4]{(x+h)^3}\right)^3+\left(\sqrt[4]{(x+h)^3}\right)^2\left(\sqrt[4]{x^3}\right)+\left(\sqrt[4]{(x+h)^3}\right)\left(\sqrt[4]{x^3}\right)^2+\left(\sqrt[4]{x^3}\right)^3}{\left(\sqrt[4]{(x+h)^3}\right)^3+\left(\sqrt[4]{(x+h)^3}\right)^2\left(\sqrt[4]{x^3}\right)+\left(\sqrt[4]{(x+h)^3}\right)\left(\sqrt[4]{x^3}\right)^2+\left(\sqrt[4]{x^3}\right)^3}\right)$$

For $\sqrt[6]{x^3}$, we need $\left(a^{\frac{6}{3}}-b^{\frac{6}{3}}\right)\left(a^{\frac{12}{3}}+a^{\frac{6}{3}}b^{\frac{6}{3}}+b^{\frac{12}{3}}\right)=\left(a^2-b^2\right)\left(a^4+a^2b^2+b^4\right)=a^{\frac{18}{3}}-b^{\frac{18}{3}}$ but we could do it more easily if we reduce first, so *p* and *q* should to be relatively prime (*p* and *q* have no common factors).

After playing with these functions and several others, we can see how all this works.

If p and q are relatively prime, we can find the derivative of the rational function $y=\sqrt[q]{x^p}$ using the definition of derivative by multiplying the difference quotient $\dfrac{\sqrt[q]{(x+h)^p}-\sqrt[q]{x^p}}{h}$ by

$$\left(\frac{\left(\sqrt[q]{(x+h)^p}\right)^{q-1}+\left(\sqrt[q]{(x+h)^p}\right)^{q-2}\left(\sqrt[q]{x^p}\right)+\left(\sqrt[q]{(x+h)^p}\right)^{q-3}\left(\sqrt[q]{x^p}\right)^2+\cdots+\left(\sqrt[q]{(x+h)^p}\right)\left(\sqrt[q]{x^p}\right)^{q-2}+\left(\sqrt[q]{x^p}\right)^{q-1}}{\left(\sqrt[q]{(x+h)^p}\right)^{q-1}+\left(\sqrt[q]{(x+h)^p}\right)^{q-2}\left(\sqrt[q]{x^p}\right)+\left(\sqrt[q]{(x+h)^p}\right)^{q-3}\left(\sqrt[q]{x^p}\right)^2+\cdots+\left(\sqrt[q]{(x+h)^p}\right)\left(\sqrt[q]{x^p}\right)^{q-2}+\left(\sqrt[q]{x^p}\right)^{q-1}}\right).$$

The result of this product will be

$$\left(\frac{1}{h}\right)\cdot\left(\frac{(x+h)^p-x^p}{\left(\sqrt[q]{(x+h)^p}\right)^{q-1}+\left(\sqrt[q]{(x+h)^p}\right)^{q-2}\left(\sqrt[q]{x^p}\right)+\left(\sqrt[q]{(x+h)^p}\right)^{q-3}\left(\sqrt[q]{x^p}\right)^2+\cdots+\left(\sqrt[q]{(x+h)^p}\right)\left(\sqrt[q]{x^p}\right)^{q-2}+\left(\sqrt[q]{x^p}\right)^{q-1}}\right)$$

which simplifies to

$$\left(\frac{1}{h}\right)\cdot\left(\frac{\binom{p}{1}(x)^{p-1}h+\binom{p}{2}(x)^{p-2}h^2+\cdots+\binom{p}{k}(x)^{p-k}h^k+\cdots+\binom{p}{p-1}(x)h^{p-1}+\binom{p}{p}h^p}{\left(\sqrt[q]{(x+h)^p}\right)^{q-1}+\left(\sqrt[q]{(x+h)^p}\right)^{q-2}\left(\sqrt[q]{x^p}\right)+\left(\sqrt[q]{(x+h)^p}\right)^{q-3}\left(\sqrt[q]{x^p}\right)^2+\cdots+\left(\sqrt[q]{(x+h)^p}\right)\left(\sqrt[q]{x^p}\right)^{q-2}+\left(\sqrt[q]{x^p}\right)^{q-1}}\right).$$

Removing the common factor of h in the numerator and denominator, we have

$$\left(\frac{\binom{p}{1}(x)^{p-1}+\binom{p}{2}(x)^{p-2}h+\cdots+\binom{p}{k}(x)^{p-k}h^{k-1}+\cdots+\binom{p}{p-1}(x)h^{p-2}+\binom{p}{p}h^{p-1}}{\left(\sqrt[q]{(x+h)^p}\right)^{q-1}+\left(\sqrt[q]{(x+h)^p}\right)^{q-2}\left(\sqrt[q]{x^p}\right)+\left(\sqrt[q]{(x+h)^p}\right)^{q-3}\left(\sqrt[q]{x^p}\right)^2+\cdots+\left(\sqrt[q]{(x+h)^p}\right)\left(\sqrt[q]{x^p}\right)^{q-2}+\left(\sqrt[q]{x^p}\right)^{q-1}}\right)$$

and taking the limit as h goes to zero, we have

$$\lim_{h\to 0}\left(\frac{\binom{p}{1}(x)^{p-1}+\binom{p}{2}(x)^{p-2}h+\cdots+\binom{p}{k}(x)^{p-k}h^{k-1}+\cdots+\binom{p}{p-1}(x)h^{p-2}+\binom{p}{p}h^{p-1}}{\left(\sqrt[q]{(x+h)^p}\right)^{q-1}+\left(\sqrt[q]{(x+h)^p}\right)^{q-2}\left(\sqrt[q]{x^p}\right)+\left(\sqrt[q]{(x+h)^p}\right)^{q-3}\left(\sqrt[q]{x^p}\right)^2+\cdots+\left(\sqrt[q]{(x+h)^p}\right)\left(\sqrt[q]{x^p}\right)^{q-2}+\left(\sqrt[q]{x^p}\right)^{q-1}}\right)=$$

$$\frac{p(x)^{p-1}}{\left(\sqrt[q]{(x)^p}\right)^{q-1}+\left(\sqrt[q]{(x)^p}\right)^{q-2}\left(\sqrt[q]{x^p}\right)+\left(\sqrt[q]{(x)^p}\right)^{q-3}\left(\sqrt[q]{x^p}\right)^2+\cdots+\left(\sqrt[q]{(x)^p}\right)\left(\sqrt[q]{x^p}\right)^{q-2}+\left(\sqrt[q]{x^p}\right)^{q-1}}=\left(\frac{p(x)^{p-1}}{q\left(\sqrt[q]{(x)^p}\right)^{q-1}}\right)$$

Finally, using laws of exponents, we find that $\dfrac{d}{dx}\sqrt[q]{x^p}=\left(\dfrac{p}{q}\right)x^{(p-1)-\left(\frac{p}{q}\right)(q-1)}=\dfrac{p}{q}x^{\frac{p}{q}-1}=\dfrac{p}{q}\sqrt[q]{x^{p-q}}$.