Calculus Challenge #10 SOLUTION

Modeling the Unreasonable Effectiveness of Guerilla Warfare

This problem builds on Challenge Problem #14 and its solution from 2007-2008. Please review that problem before beginning your work.

http://courses.ncssm.edu/math/POW/POW07_08/Calculus%20Challenge%20%2314%20SOLUTION.pdf

As described in Problem 14 from 2007-2008, the defining equations for the direct fire combat model are very simple, $\frac{dA}{dt} = -bB$ and $\frac{dB}{dt} = -aA$. However, suppose one of the two combatants, we will call them *G*, is fighting a guerilla style battle while the other *A* is fighting a conventional battle. Since group *A* is still fighting a conventional directed fire battle, we have $\frac{dA}{dt} = -gG$. But what about group *G*? How will their defining equation be altered by this restructuring of the conflict?

A guerilla style battle is also called an *area fire* (as opposed to *direct fire*) model. Instead of being exposed, the guerilla fighters are hidden. Their opponents can't see them to fire directly at them. All that is known is that the guerilla contingent is "over in that field". So, they fire into that area and hope for a hit. If a group of *G* combatants is fighting guerilla style, what is $\frac{dG}{dt}$? Naturally, $\frac{dG}{dt}$ depends on the effectiveness and size of the opposing force as before, but what would improve the chances of the opposing force making a hit as it fires into the area holding the guerilla force? The larger the size of *G*, the more likely they will take a loss by a chance hit.

So, the change in the guerilla force over time can be modeled by $\frac{dG}{dt} = -aAG$.

In this setting, the coefficient of combat effectiveness *a* must also take into account the style of fighting, since the probability of a hit is dramatically reduced from the direct fire model. In general, the value of *a* is much smaller (by factors of 100 to 1000 depending on the terrain and type of weapons being used) than the coefficients in directed fire equations.

A battle between a force fighting conventional style and a force fighting guerilla style would be modeled by the coupled differential equations

$$\frac{dA}{dt} = -g \cdot G \quad \text{and} \quad \frac{dG}{dt} = -a\,AG.$$

1. Find *A* as a function of *G* by solving a differential equation involving $\frac{dA}{dG}$. Show that

$$A = \sqrt{2\frac{g}{a}G + A_0^2 - 2\frac{g}{a}G_0} \quad \text{and} \quad G = -\frac{aA^2}{2g} + \frac{aA_0^2}{2g} - G_0.$$

If $\frac{dA}{dt} = -g \cdot G$ and $\frac{dG}{dt} = -a\,AG$, then $\frac{\frac{dA}{dt}}{\frac{dG}{dt}} = \frac{-g \cdot G}{-(a\,A) \cdot G}$, which we rewrite as $\frac{dA}{dG} = \frac{g}{(a\,A)}$.

Separating variables, we have $\int aA\, dA = \int g\; dG$. Integrating, we find that

$$\frac{aA^2}{2} = gG + c\,.$$

At the beginning of the battle we have A_0 and G_0 combatants, so $c = \frac{aA_0^2}{2} - gG_0$.

This gives $\frac{aA^2}{2} = gG + \frac{aA_0^2}{2} - gG_0$. Solving for *A* in terms of *G* and then for *G* in terms of *A*, we find that

$$A = \sqrt{2\frac{g}{a}G + A_0^2 - 2\frac{g}{a}G_0} \text{ and } G = \frac{aA^2}{2g} - \frac{aA_0^2}{2g} + G_0\,.$$

(Note: This second equation is opposite in sign of that expected. Evidently, the problem poser made a mistake in his work.)

2. We can define a "fair fight" to be a battle in which the two forces reach zero at the same time. That is, the starting conditions for which the battle would end with *A* = *G* = 0. Use the constant in the equation from (1) to define a "fair fight".

If $A = \sqrt{2\frac{g}{a}G + A_0^2 - 2\frac{g}{a}G_0}$, then when *A* = *G* = 0, $\frac{aA_0^2}{2} = gG_0$. This means that a guerilla force of G_0 is in a fair fight with a classical force of $\frac{aA_0^2}{2g}$. If $G_0 > \frac{aA_0^2}{2g}$, then *G* will win and if $G_0 < \frac{aA_0^2}{2g}$ then *A* will win.

3. According to mathematician Courtney Coleman, during the Vietnam War, the ratio $\frac{a}{2g}$ was estimated at $\frac{1}{1000}$. Your result from (2) should indicate that a group of $G_0 = 10$ guerillas could be effective against a force 10 times their size fighting a conventional battle. Fill in the table below to illustrate how the effectiveness of guerilla warfare is reduced as the size of the guerilla force increases.

G_0	10	20	30	50	100	1000	1200
A_0	100	141	173	224	316	1000	1095
Ratio	10	7.1	5.8	4.5	3.2	1	0.9

Comparing the size of the Guerilla force G_0 with A_0 in a fair fight

4. We can find $A(t)$, but we have to be a bit clever to do it. We know that $G = \frac{aA^2}{2g} - \frac{aA_0^2}{2g} + G_0$ and further, that $\frac{dA}{dt} = -gG$. Use these two equations to find $A(t)$. Does the sign of $\frac{aA_0^2}{2g} - G_0$ affect the method of integration and the final result? Explain why or why not.

$\frac{dA}{dt} = -gG = \frac{-aA^2}{2} + \frac{aA_0^2}{2} - gG_0$. To make the work that follows a little easier, we will write this as $\frac{dA}{dt} = \frac{-a}{2}\left(A^2 - A_0^2 + \frac{2g}{a}G_0\right)$ or So, $\frac{dA}{dt} = -k\left(A^2 + c\right)$ with $k = \frac{a}{2}$ and $c = \frac{2g}{a}G_0 - A_0^2$.

Can we solve the differential equation $\frac{dA}{dt} = -k\left(A^2 + c\right)$?

If we separate variables, we have $\int \frac{dA}{A^2 + c} = \int -k\,dt$. The integration of the right side is straight-forward, but the left side depends upon the sign of *c*. Recall that, if $G_0 > \frac{aA_0^2}{2g}$, then *A* will win and if $G_0 < \frac{aA_0^2}{2g}$ then *G* will win. So, the solutions will differ depending upon whether *A* wins or loses the battle. This should make sense, since the curves for a losing fight should differ from that of a winning fight.

If $G_0 > \frac{aA_0^2}{2g}$, then $c = \frac{2g}{a}G_0 - A_0^2 > 0$. When the value of *c* is positive, this represents *A* winning the battle.

If $c > 0$, then $\int \frac{dA}{A^2 + c} = \frac{1}{c}\int \frac{dA}{\left(\frac{A}{\sqrt{c}}\right)^2 + 1}$ involves an inverse tangent integral. Let $u = \frac{A}{\sqrt{c}}$ so $\sqrt{c}\,du = dA$. Then $\int \frac{dA}{A^2 + c} = \frac{1}{\sqrt{c}}\int \frac{du}{u^2 + 1} = \frac{1}{\sqrt{c}}\tan^{-1}\left(\frac{A}{\sqrt{c}}\right)$.

So, $\int \frac{dA}{A^2 + c} = \int -k\,dt$ becomes $\frac{1}{\sqrt{c}}\tan^{-1}\left(\frac{A}{\sqrt{c}}\right) = -kt + q$. Solving for *A*, we find that $\tan^{-1}\left(\frac{A}{\sqrt{c}}\right) = -k\sqrt{c}t + q$, so $q = \tan^{-1}\left(\frac{A_0}{\sqrt{c}}\right)$ and

$$A = \sqrt{c}\tan\left(-k\sqrt{c}t + \tan^{-1}\left(\frac{A_0}{\sqrt{c}}\right)\right).$$

Since $k=\frac{a}{2}$ and $c=\frac{2g}{a}G_0-A_0^2$, we have

$$A=\sqrt{\frac{2g}{a}G_0-A_0^2}\,\tan\left(\frac{-a}{2}\sqrt{\frac{2g}{a}G_0-A_0^2}\,t+\tan^{-1}\left(\frac{A_0}{\sqrt{\frac{2g}{a}G_0-A_0^2}}\right)\right)$$

This can be rewritten using the tangent sum formula if desired as

$$A=\sqrt{\frac{2g}{a}G_0-A_0^2}\left(\frac{\left(\frac{A_0}{\sqrt{\frac{2g}{a}G_0-A_0^2}}\right)-\tan\left(\frac{a}{2}\sqrt{\frac{2g}{a}G_0-A_0^2}\,t\right)}{1+\left(\frac{A_0}{\sqrt{\frac{2g}{a}G_0-A_0^2}}\right)\tan\left(\frac{a}{2}\sqrt{\frac{2g}{a}G_0-A_0^2}\,t\right)}\right).$$

If, however, $G_0<\frac{aA_0^2}{2g}$, then $c=\frac{2g}{a}G_0-A_0^2<0$ and A loses the battle. When the value of c is negative, we no longer have an inverse tangent integral.

If $c<0$, then let $c=-C$, so $\int\frac{dA}{A^2-C}$ and we can factor the denominator and use partial fractions.

So, $\frac{1}{A^2-C}=\frac{u}{A+\sqrt{C}}+\frac{v}{A-\sqrt{C}}$ and $(u+v)A+(v-u)\sqrt{C}=1$. So, $v=-u$ and $v=\frac{1}{2\sqrt{C}}$, $u=\frac{-1}{2\sqrt{C}}$.

So, $\int\frac{dA}{A^2-C}=\frac{1}{2\sqrt{C}}\int\frac{1}{A-\sqrt{C}}-\frac{1}{A+\sqrt{C}}\,dA=\frac{1}{2\sqrt{C}}\left(\ln\left(\frac{A-\sqrt{C}}{A+\sqrt{C}}\right)\right)$.

Now, $\frac{1}{2\sqrt{C}}\left(\ln\left(\frac{A-\sqrt{C}}{A+\sqrt{C}}\right)\right)=-kt+p$ and $\left(\ln\left(\frac{A-\sqrt{C}}{A+\sqrt{C}}\right)\right)=-2\sqrt{C}kt+p$, so $\left(\frac{A-\sqrt{C}}{A+\sqrt{C}}\right)=Pe^{-2\sqrt{C}kt}$. So, $P=\left(\frac{A_0-\sqrt{C}}{A_0+\sqrt{C}}\right)$ Solving for *A*, we find $A\left(1-Pe^{-2\sqrt{C}kt}\right)=\sqrt{C}\left(1+Pe^{-2\sqrt{C}kt}\right)$, so $A=\frac{\sqrt{C}\left(1+Pe^{-2\sqrt{C}kt}\right)}{\left(1-Pe^{-2\sqrt{C}kt}\right)}$. Substituting for *C*, *P*, and *k*, we have

$$A=\frac{\sqrt{A_0^2-\frac{2g}{a}G_0}\left(1+\left(\frac{A_0-\sqrt{A_0^2-\frac{2g}{a}G_0}}{A_0+\sqrt{A_0^2-\frac{2g}{a}G_0}}\right)e^{-a\sqrt{A_0^2-\frac{2g}{a}G_0}t}\right)}{\left(1-\left(\frac{A_0-\sqrt{A_0^2-\frac{2g}{a}G_0}}{A_0+\sqrt{A_0^2-\frac{2g}{a}G_0}}\right)e^{-a\sqrt{A_0^2-\frac{2g}{a}G_0}t}\right)}$$

when *A* wins the battle.

5. Find $G(t)$. We will similarly have two different curves for *G*. One for when *G* wins and another for when *G* loses.

When *A* wins (*G* loses) we have $A=\sqrt{\frac{2g}{a}G_0-A_0^2}\tan\left(\frac{-a}{2}\sqrt{\frac{2g}{a}G_0-A_0^2}t+\tan^{-1}\left(\frac{A_0}{\sqrt{\frac{2g}{a}G_0-A_0^2}}\right)\right)$,

so if $G=-\frac{aA^2}{2g}+\frac{aA_0^2}{2g}-G_0$, then by substitution, we know that

$$G=\frac{a\left[\sqrt{\frac{2g}{a}G_0-A_0^2}\tan\left(\frac{-a}{2}\sqrt{\frac{2g}{a}G_0-A_0^2}t+\tan^{-1}\left(\frac{A_0}{\sqrt{\frac{2g}{a}G_0-A_0^2}}\right)\right)\right]^2}{2g}-\frac{aA_0^2}{2g}+G_0.$$

Also, when A loses (G wins) we have $A = \dfrac{\sqrt{A_0^2 - \frac{2g}{a}G_0}\left(1 + \left(\dfrac{A_0 - \sqrt{A_0^2 - \frac{2g}{a}G_0}}{A_0 + \sqrt{A_0^2 - \frac{2g}{a}G_0}}\right)e^{-a\sqrt{A_0^2 - \frac{2g}{a}G_0}\,t}\right)}{\left(1 - \left(\dfrac{A_0 - \sqrt{A_0^2 - \frac{2g}{a}G_0}}{A_0 + \sqrt{A_0^2 - \frac{2g}{a}G_0}}\right)e^{-a\sqrt{A_0^2 - \frac{2g}{a}G_0}\,t}\right)}$.

Again, by substitution in

$$G = -\frac{a\left[\dfrac{\sqrt{A_0^2 - \frac{2g}{a}G_0}\left(1 + \left(\dfrac{A_0 - \sqrt{A_0^2 - \frac{2g}{a}G_0}}{A_0 + \sqrt{A_0^2 - \frac{2g}{a}G_0}}\right)e^{-a\sqrt{A_0^2 - \frac{2g}{a}G_0}\,t}\right)}{\left(1 - \left(\dfrac{A_0 - \sqrt{A_0^2 - \frac{2g}{a}G_0}}{A_0 + \sqrt{A_0^2 - \frac{2g}{a}G_0}}\right)e^{-a\sqrt{A_0^2 - \frac{2g}{a}G_0}\,t}\right)}\right]^2}{2g} + \frac{aA_0^2}{2g} - G_0.$$

6. The consequence of having small guerilla units ($G_0 = 10$) in Vietnam is that the total armed forces of the US and South Vietnamese combined needed to be about 10 times that of the Viet Cong and North Vietnamese for the US and South Vietnamese to have a reasonable chance of victory. In 1968, the ratio of forces was at 6 to 1. General Westmoreland asked for an additional 206,000 troops, but this request was rejected by President Johnson. Would the additional troops have turned the tide of the war? The table below gives the estimated number of forces in Vietnam at the time of the request.

Conventional Forces			Guerrilla Forces	
American	510,000		North Vietnamese	50,000
South Vietnamese: Regulars	600,000		Viet Cong	230,000
South Vietnamese: local defense	500,000			
Other Allies	70,000			
Total	1,680,000			280,000

The ratio of troops at the time of the request was $\dfrac{1{,}680{,}000}{280{,}000} \approx \dfrac{6}{1}$. With the additional 206,000 US troops, the ratio would have been raised to $\dfrac{1{,}866{,}000}{280{,}000} \approx \dfrac{6.7}{1}$. This was not nearly enough to affect the outcome of the conflict. To truly be productive, the US needed to increase its troop level from 510,00 to 1,630,000. Clearly, an impossible number.   According to Courtney Coleman of Harvey Mudd College, "it was analyses such as this, coupled with the disquiet of the American people about the whole affair, that led President Johnson to seek a political solution to the Vietnamese conflict. He rejected Westmoreland's request and initiated the Paris peace talks, which eventually led to the American disengagement in 1973."

Reference: Coleman, Courtney S. "Combat Models", in *Differential Equation Models*, Martin Braun, Courtney Coleman, and Donald Drew, Editors, Vol 1 of *Models in Applied Mathematics*, William Lucas, Editor, Springer-Verlag, NY, 1983.