Calculus Challenge #11 Solutions due April 14

Note: Since many schools will be on vacation sometime during the next few week, you have three weeks for this problem. We will take a break as you begin your review for the AP exam. The final problem for this year will appear on May 12.

Last week, an article on "The Mathematics of Cancer" appeared in Forbes magazine. The entire article is at http://www.forbes.com/forbes/2010/0315/opinions-health-cancer-larry-norton-ideas-opinions.html?partner=email An excerpt, which suggests this bi-week's problem is given below.

From Forbes Magazine, March 15, 2010 ***The Mathematics Of Cancer***, by ***Robert Langreth***

Larry Norton sees some of the toughest cases as deputy physician-in-chief for breast cancer at Memorial Sloan-Kettering Cancer Center. He has access to the most advanced imaging machines, the best surgeons and numerous new tumor-fighting drugs. But often the fancy technology helps only temporarily. Sometimes a big tumor will shrink dramatically during chemotherapy. Then all of a sudden it comes back in seven or eight locations simultaneously.

Norton thinks adding more mathematics to the crude science of cancer therapy will help. He says that oncologists need to spend much more time devising and analyzing equations that describe how fast tumors grow, how quickly cancer cells develop resistance to therapy and how often they spread to other organs. By taking such a quantitative approach, researchers may be able to create drug combinations that are far more effective than the ones now in use. "I have a suspicion that we are using almost all the cancer drugs in the wrong way," he says. "For all I know, we may be able to cure cancer with existing agents."

His strategy is unusual among cancer researchers, who have tended to focus on identifying cancer-causing genes rather than writing differential equations to describe the rate of tumor spread. Yet adding a dose of numbers has already led to important changes in breast cancer treatment. The math of tumor growth led to the discovery that just changing the frequency of chemo treatments can boost their effect significantly…

Ever since he was a fellow at the National Cancer Institute in the 1970s he has been trying to come up with mathematical laws that describe tumor growth. He treated a lymphoma patient whose tumor shrank rapidly during chemotherapy. A year later the cancer returned worse than ever. The speed with which the tumor grew back didn't jibe with the prevailing notion that most tumors grew in a simple exponential fashion.

Working with NCI statistician Richard Simon, Norton came up with a new model of tumor growth based on the work of the 19th-century mathematician Benjamin Gompertz. The concept (which other researchers proposed in the 1960s) holds that tumor growth generally follows an S-shape curve. Microscopic tumors below a certain threshold barely grow at all. Small tumors grow exponentially, but the rate of growth slows dramatically as tumors get bigger, until it reaches a plateau. A corollary of this: The faster you shrink a tumor with chemo, the quicker it will grow back if you haven't killed it all.

Based on these rates of growth, Norton argued that giving the same total dose of chemotherapy over a shorter period of time would boost the cure rate by limiting the time tumors could regrow between treatments. The concept got a skeptical reaction initially. "People said it was a total waste of time," he recalls. It took decades before Norton was able to prove his theory. But in 2002 a giant government trial showed that giving chemotherapy every two weeks instead of every three lowered the risk of breast cancer recurrence by 26% over three years, even though the two groups got the same cumulative dose…

I thought it would be interesting to look into the Gompertz model noted in the article.

In making observations on the growth of animal tumors, mathematician Benjamin Gompertz found that the size y of the tumor seemed to obey the differential equation

$$\frac{dy}{dt} = -ky \ln\left(\frac{y}{b}\right)$$

with k and b positive constants.

Using the Gompertz model, answer the following questions.

a) In this model of tumor growth, what is the range of y? What does a slope field tell you about solutions to this equation?

b) At what size is the tumor growing most rapidly?

c) Solve the differential equation with the initial condition $y(0) = y_0$. At what time is the tumor growing most rapidly?

d) Find a relation between y_0, k, and b so that the graph of y vs t has no point of inflection.

e) Find $\lim_{t \to \infty} y(t)$. What does this say about the growth of the tumor?

f) The Gompertz model is sometimes defined by the differential equation $\frac{dy}{dt} = me^{-ht}y$. What is the relationship between k and b in the first model and m and h in this model?

g) Which differential equation tells you more about the pattern of growth of the tumor?