Calculus Challenge #12 **SOLUTION**

One of the topics you studied this year involved conditionally and absolutely convergent series. We will look at what it means to be conditionally convergent in this challenge.

1. Given that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ whenever $|x| < 1$, prove that

$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots$. **(Equation 1)**

The series in Equation 1 is known as the alternating harmonic series.

We know the sum of an infinite geometric series $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ whenever $|x| < 1$. So, by substitution, we have $1 + (-x) + (-x)^2 + (-x)^3 + \cdots = \frac{1}{1-(-x)}$ whenever $|(-x)| < 1$ or $1 - x + x^2 - x^3 + x^4 + \cdots = \frac{1}{1+x}$ whenever $|x| < 1$.

Integrating, we have $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots + C = \ln(1+x)$ whenever $|x| < 1$. To find the constant of integration, we let $x = 0$, so $C = 0$. Then $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots = \ln(1+x)$ whenever $|x| < 1$.

However, we recall that while series created by integration have the same radius of convergence as the original, they may include the endpoints of the interval that were not part of the interval of convergence for the original. In this example, 1 is in the interval of convergence, which is (-1, 1] since the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$ converges by the alternating series test.

So

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots = \ln(2).$$

2. Given $\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots$, suppose we rewrite each positive term $\frac{1}{k}$ on the right side of the equation as $\left(\frac{2}{k} - \frac{1}{k}\right)$. The sum would still, of course, be $\ln(2)$.

$$\ln(2) = (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} + \left(\frac{2}{9} - \frac{1}{9}\right) - \frac{1}{10} + \cdots \qquad \textbf{(Equation 2)}$$

Now, divide both sides of Equation (2) by 2. The sum of the new series is, naturally, $\frac{1}{2}\ln(2)$. What do you notice about this "new" series when compared to the right side of Equation 1?

If we re-write, we have

$$\ln(2) = [2-1] - \frac{1}{2} + \left[\frac{2}{3} - \frac{1}{3}\right] - \frac{1}{4} + \left[\frac{2}{5} - \frac{1}{5}\right] - \frac{1}{6} + \left[\frac{2}{7} - \frac{1}{7}\right] - \frac{1}{8} + \left[\frac{2}{9} - \frac{1}{9}\right] - \frac{1}{10} + \cdots$$

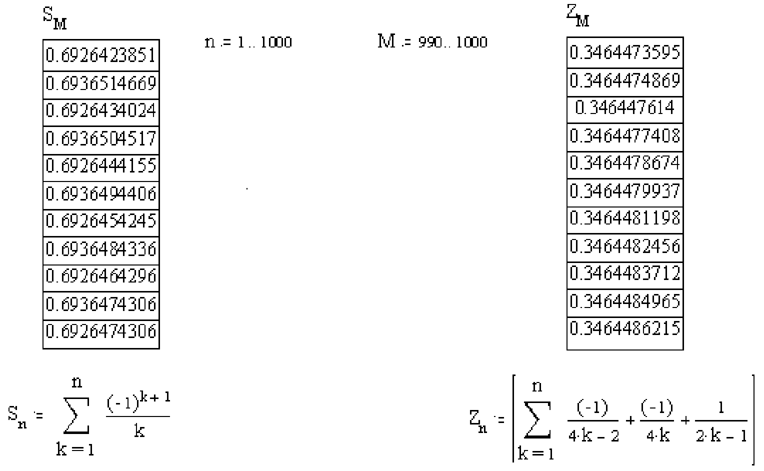
Now, divide both sides by 2. The result is

$$\frac{\ln(2)}{2} = \left[1 - \frac{1}{2}\right] - \frac{1}{4} + \left[\frac{1}{3} - \frac{1}{6}\right] - \frac{1}{8} + \left[\frac{1}{5} - \frac{1}{10}\right] - \frac{1}{12} + \left[\frac{1}{7} - \frac{1}{14}\right] - \frac{1}{16} + \left[\frac{1}{9} - \frac{1}{18}\right] - \frac{1}{20} + \cdots.$$

Removing the parentheses, we have

$$\frac{\ln(2)}{2} = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \frac{1}{9} - \frac{1}{18} - \frac{1}{20} + \cdots$$

The summands in this series are **exactly** the same terms as in the alternating harmonic series. The only difference is in the order in which they are added. Yet one adds to ln(2) and the other to $\frac{1}{2}\ln(2)$. A short computer program to sum the two series will convince you the sums differ.



$n := 1..1000$ $\qquad$ $M := 990..1000$

S_M	Z_M
0.6926423851	0.3464473595
0.6936514669	0.3464474869
0.6926434024	0.346447614
0.6936504517	0.3464477408
0.6926444155	0.3464478674
0.6936494406	0.3464479937
0.6926454245	0.3464481198
0.6936484336	0.3464482456
0.6926464296	0.3464483712
0.6936474306	0.3464484965
0.6926474306	0.3464486215

$$S_n := \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$$

$$Z_n := \left|\sum_{k=1}^{n} \frac{(-1)}{4\cdot k - 2} + \frac{(-1)}{4\cdot k} + \frac{1}{2\cdot k - 1}\right|$$

We see the partials sums of the harmonic series on the left (converging to ln(2)) and the partial sums of the rearranged series (Taking three terms at a time).

3. Multiply Equation 1 by $\frac{1}{2}$ and add to Equation 1. The left side is $\tfrac{3}{2}\ln(2)$. What do you notice about this "new" series on the right side of the equation?

$\tfrac{1}{2}\ln(2)=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots\right)=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}+\cdots.$

If we now add this series to $\ln(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots$, notice that the new series is

$$\tfrac{3}{2}\ln(2)=1+\left(\frac{1}{2}-\frac{1}{2}\right)+\frac{1}{3}-\left(\frac{1}{4}+\frac{1}{4}\right)+\frac{1}{5}+\left(\frac{1}{6}-\frac{1}{6}\right)+\frac{1}{7}-\left(\frac{1}{8}+\frac{1}{8}\right)+\cdots.$$

But this is just $\tfrac{3}{2}\ln(2)=1+\frac{1}{3}-\frac{1}{2}-+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots.$

Notice the terms in this series are the same as in the alternating harmonic only in a new order.

Note: This **does not** mean that

$$\tfrac{3}{2}\ln(2)=1+\frac{1}{3}-\frac{1}{2}-+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\cdots=1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots=\tfrac{1}{2}\ln(2).$$

The series and the sums are different even though the terms in the sums are the same.

4. Prove that the subseries of only positive terms $P_n=1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\cdots$ and the subseries of only negative terms $N_n=-\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{6}+\cdots$ both diverge.

The two sub-series of negative terms, $N_n=-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\cdots$, and positive terms, $P_n=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$, are both divergent series. We can see that N_n diverges, since $N_n=-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\cdots=-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$ which is a constant multiplied by the sum of the harmonic series which diverges. Since N_n diverges, so does $-N_n=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots$, which can be compared to $P_n=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$. Each term of P_n is larger than the corresponding term of $-N_n$, so the divergence of $-N_n$ guarantees the divergence of P_n. Since each sub-series diverges, we can rearrange the series to converge to any number we choose.

5. If the subseries of positive and negative terms both diverge, but the series itself converges, then we have a conditionally convergent series. Be careful how you arrange the terms when you add. The sum of the series depends on the order of addition. Why doesn't this violate the commutative property of addition?

The commutative property of addition states that two addends can be reordered without changing the sum. This can be extended to any finite number of addends. However, as our example shows, it does not extend to an infinite number of addends. Not all infinite sums can be rearranged to create different sums. For example, no matter how you rearrange the terms of $\frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$ the sum will always be $\frac{1}{e}$.

6. Find a pattern of *p* positive terms and *n* negative terms that sum to zero.

To create a rearrangement that sums to 0, start with 1 and subtract enough terms until the partial sum is negative, then add until positive, subtract until negative, etc.

$$\begin{aligned}
&1 > 0\\
&1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} < 0\\
&1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} > 0\\
&1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} < 0
\end{aligned}$$

If we continue in this fashion, we see that 1 positive term and 4 negative terms always results in a change of sign.

7. Explain how you can rearrange the terms in the alternating harmonic series so they will sum to π.

To rearrange the series to sum to π, for example, take enough terms from P_n to be just larger than π. Say $P_k < \pi < P_{k+1}$. Now, add the first term of N_n. $P_{k+1} + N_1 < \pi$. Now add enough additional terms from P_n until the new sum is just larger than π. Say $P_m + N_1 < \pi < P_{m+1} + N_1$. Now, add the second term of N_n. $P_{m+1} + N_1 + N_2 < \pi$. Since each new term is smaller than the previous, the size of the jumps on either side of π get smaller and smaller. If we continue this process, the resulting series converges to π.

Hickman High School used the following approach, based on the result in Question 8.

If $\pi = \left(\ln(2) + \frac{1}{2}\ln\left(\frac{p}{n}\right) \right) \rightarrow 2\pi - 2\ln(2) = \ln\left(\frac{p}{n}\right) \rightarrow 133.8729 = \frac{p}{n}$, so $133.8729 \approx \frac{3346}{25}$.

Then *P* equals 3346 and *n* equals 25. You can roughly get a sum of π with 3345 positive terms followed by 25 negative terms, and so on…

8. We can derive the value of the sum of a rearrangement of the alternating harmonic series if the rearrangement is consistent, that is, *p* positive terms followed by *n* negative terms, repeating. First, we need three pieces of information about the *harmonic* series.

- First, we can write the first 2*N* terms of the *harmonic* series, H_{2N}, as the sum on *N* odd terms and *N* even terms. That is, $H_{2N} = O_N + E_N$.
- Second, we need to recognize that $E_N = \frac{1}{2} H_N$, since $E_N = \sum_{n=1}^{N} \frac{1}{2n}$ and $H_N = \sum_{n=1}^{N} \frac{1}{n}$.
- Third, the difference in the sum of the first *N* terms of the *harmonic* series and $\ln(N)$ converges to a constant called Euler's number. Euler's number is often symbolized by γ.

Using these three pieces of information, we can determine the value to which adding *p* positive terms and *n* negative terms of the *alternating harmonic* series converges. We need to cleverly add zero twice! Remember, the odd terms are positive and the even terms are negative in the alternating harmonic series.

$$\text{Let } S = \lim_{k\to\infty}\left(O_{kp} - E_{kn}\right).$$

For partial sums, we have *k* groups of *p* positives (odds) and *n* negatives (evens). For example, in Equation 2 we show *k* = 5, *p* = 1, and *n* = 2 written out.

Rewrite *S* as

$$S_k = O_{kp} - E_{kn}, \text{ so } S_k = O_{kp} + \left(E_{kp} - E_{kp}\right) - E_{kn}.$$

Rearranging, we have

$$S_k = \left(O_{kp} + E_{kp}\right) - E_{kp} - E_{kn} = H_{2kp} - \frac{1}{2}H_{kp} - \frac{1}{2}H_{kn}$$

using the first two ideas above. Now, compare each of the three harmonic series in the expression above the value of the associated logarithm.

$$S_k = \left(H_{2kp} - \ln(2kp)\right) - \frac{1}{2}\left(H_{kp} - \ln(kp)\right) - \frac{1}{2}\left(H_{kn} - \ln(kn)\right) + \left(\ln(2kp) - \frac{1}{2}\ln(kp) - \frac{1}{2}\ln(kn)\right).$$

Now, take the limit as $k \to \infty$.

To what number does the sum of *p* consecutive positive (odd) terms plus *n* consecutive negative (even) terms from the alternating geometric series converge?

If $S_k = \left(H_{2kp} - \ln(2kp)\right) - \frac{1}{2}\left(H_{kp} - \ln(kp)\right) - \frac{1}{2}\left(H_{kn} - \ln(kn)\right) + \left(\ln(2kp) - \frac{1}{2}\ln(kp) - \frac{1}{2}\ln(kn)\right)$,

then

$$S_k = \left(H_{2kn} - \ln(2kn)\right) - \frac{1}{2}\left(H_{kn} - \ln(kn)\right) - \frac{1}{2}\left(H_{km} - \ln(km)\right) + \ln(2kn) - \frac{1}{2}\ln(kn) - \frac{1}{2}\ln(km).$$

Now, take the limit as $k \to \infty$.

As $k \to \infty$, $\left(H_{2kn} - \ln(2kn)\right) \to \gamma$, $\left(H_{kn} - \ln(kn)\right) \to \gamma$, and $\left(H_{km} - \ln(km)\right) \to \gamma$, so we have

$$S = \gamma - \frac{1}{2}\gamma - \frac{1}{2}\gamma + \lim_{k\to\infty}\left[\ln(2kn) - \frac{1}{2}\ln(kn) - \frac{1}{2}\ln(km)\right].$$

But this last limit simplifies to

$$S = \lim_{k\to\infty}\left[\ln(2) + \ln(kn) - \frac{1}{2}\ln(kn) - \frac{1}{2}\ln(km)\right] = \lim_{k\to\infty}\left[\ln(2) + \frac{1}{2}\ln(kn) - \frac{1}{2}\ln(km)\right] = \lim_{k\to\infty}\left[\ln(2) + \frac{1}{2}\ln\left(\frac{n}{m}\right)\right]$$

which is just

$$S = \ln(2) + \frac{1}{2}\ln\left(\frac{n}{m}\right).$$

This formula gives the value $S = \frac{1}{2}\ln(2)$ when $n = 1$ and $m = 2$. Notice that the value of S is zero if $n = 1$ and $m = 4$.

So when can we rearrange the terms in an infinite series? It turns out that, if a series is only conditionally convergent, as is the series for $\ln(2)$, we cannot arbitrarily rearrange an infinite number of terms the terms without possibly altering the value of the series. For series that are absolutely convergent, altering the order of the terms does not affect the sum. For series that are conditionally convergent, the terms can be rearranged to form a series that converges to any chosen value, since a conditionally convergent series consists of a divergent sub-series of positive terms and a divergent sub-series of negative terms.