Calculus Challenge #12

Solutions Due May 26

One of the topics you studied this year involved conditionally and absolutely convergent series. We will look at what it means to be conditionally convergent in this challenge.

1. Given that $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ whenever $|x| < 1$, prove that

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots. \qquad \textbf{(Equation 1)}$$

The series in Equation 1 is known as the alternating harmonic series.

2. Given $\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots$, suppose we rewrite each positive term $\dfrac{1}{k}$ on the right side of the equation as $\left(\dfrac{2}{k} - \dfrac{1}{k}\right)$. The sum would still, of course, be $\ln(2)$.

$$\ln(2) = (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} + \left(\frac{2}{9} - \frac{1}{9}\right) - \frac{1}{10} + \cdots \qquad \textbf{(Equation 2)}$$

Now, divide both sides of Equation (2) by 2. The sum of the new series is, naturally, $\frac{1}{2}\ln(2)$. What do you notice about this "new" series when compared to the right side of Equation 1?

3. Multiply Equation 1 by $\dfrac{1}{2}$ and add to Equation 1. The left side is $\frac{3}{2}\ln(2)$. What do you notice about this "new" series on the right side of the equation?

4. Prove that the subseries of only positive terms $P_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ and the subseries of only negative terms $N_n = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \cdots$ both diverge.

5. If the subseries of positive and negative terms both diverge, but the series itself converges, then we have a conditionally convergent series. Be careful how you arrange the terms when you add. The sum of the series depends on the order of addition. Why doesn't this violate the commutative property of addition?

6. Find a pattern of p positive terms and n negative terms that sum to zero.

7. Explain how you can rearrange the terms in the alternating harmonic series so they will sum to π.

8. We can derive the value of the sum of a rearrangement of the alternating harmonic series if the rearrangement is consistent, that is, *p* positive terms followed by *n* negative terms, repeating. First, we need three pieces of information about the *harmonic* series.

- First, we can write the first 2*N* terms of the *harmonic* series, H_{2N}, as the sum on *N* odd terms and *N* even terms. That is, $H_{2N} = O_N + E_N$.
- Second, we need to recognize that $E_N = \frac{1}{2} H_N$, since $E_N = \sum_{n=1}^{N} \frac{1}{2n}$ and $H_N = \sum_{n=1}^{N} \frac{1}{n}$.
- Third, the difference in the sum of the first *N* terms of the *harmonic* series and $\ln(N)$ converges to a constant called Euler's number. Euler's number is often symbolized by γ.

Using these three pieces of information, we can determine the value to which adding *p* positive terms and *n* negative terms of the *alternating harmonic* series converges. We need to cleverly add zero twice! Remember, the odd terms are positive and the even terms are negative in the alternating harmonic series.

$$\text{Let } S = \lim_{k \to \infty} \left(O_{kp} - E_{kn} \right).$$

For partial sums, we have *k* groups of *p* positives (odds) and *n* negatives (evens). For example, in Equation 2 we show *k* = 5, *p* = 1, and *n* = 2 written out.

Rewrite *S* as

$$S_k = O_{kp} - E_{kn}, \text{ so } S_k = O_{kp} + \left(E_{kp} - E_{kp} \right) - E_{kn}.$$

Rearranging, we have

$$S_k = \left(O_{kp} + E_{kp} \right) - E_{kp} - E_{kn} = H_{2kp} - \frac{1}{2} H_{kp} - \frac{1}{2} H_{kn}$$

using the first two ideas above. Now, compare each of the three harmonic series in the expression above the value of the associated logarithm.

$$S_k = \left(H_{2kp} - \ln(2kp) \right) - \frac{1}{2}\left(H_{kp} - \ln(kp) \right) - \frac{1}{2}\left(H_{kn} - \ln(kn) \right) + \left(\ln(2kp) - \frac{1}{2}\ln(kp) - \frac{1}{2}\ln(kn) \right).$$

Now, take the limit as $k \to \infty$.

To what number does the sum of *p* consecutive positive (odd) terms plus *n* consecutive negative (even) terms from the alternating geometric series converge?