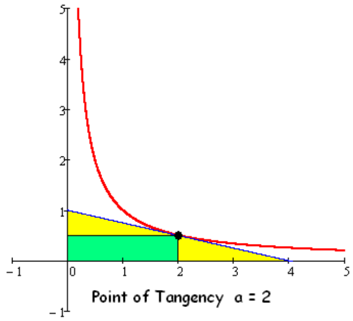
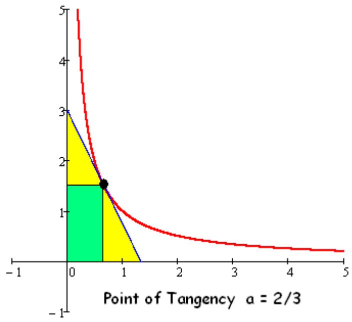
Calculus Challenge #2

Solution due: November 4, 2009

Consider the portion of the graph of $y = \frac{1}{x}$ in the first quadrant.





1. Show that the segment of the tangent to the curve in the first quadrant is bisected at the point of contact. This is true for every point of tangency (*x* = *a*).

The equation of the tangent at *x* = *a* is $T(x) = f(a) + f'(a)(x-a)$. In this example, we have $T(x) = \frac{1}{a} - \frac{1}{a^2}(x-a) = \frac{-x}{a^2} + \frac{2}{a}$. The coordinates of the point of tangency are $\left(a, \frac{1}{a}\right)$.

The intercepts (endpoints of the segment in the first quadrant) are $\left(0, \frac{2}{a}\right)$ and $(2a, 0)$.

The length of this segment is $\sqrt{(2a-0)^2 + \left(0-\frac{2}{a}\right)^2} = \sqrt{4a^2 + \frac{4}{a^2}} = \frac{2}{a}\sqrt{a^4+1}$.

The midpoint of this segment is $\left(\frac{0+2a}{2}, \frac{\frac{2}{a}+0}{2}\right) = \left(a, \frac{1}{a}\right)$, so the point of tangency is always the midpoint of the segment.

2. How does the area of the triangle bounded by the coordinate axes and the tangent segment vary with the point of tangency?

Since the intercepts are $\left(0, \frac{2}{a}\right)$ and $(2a, 0)$ The area of the triangle is always $\frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$. Regardless of the point of tangency, the area of the triangle defined by the tangent is constant. We call this an invariant. Looking for invariants is a common tool in mathematical investigations.

3. Find the point of tangency, $x = a$, at which the ratio of the area of the rectangle with opposite vertices at (0, 0) and at $\left(a, \frac{1}{a}\right)$ to the area of the triangle defined in 2) is maximum.

The area of the rectangle is always $a\left(\frac{1}{a}\right) = 1$. Since the area of the triangle is fixed as well, the ratio is always ½ no matter which tangent we choose. There is no location for which this ratio is a maximum.

4. If we alter the function to $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$, $y = \frac{1}{\sqrt{x}}$, or in general, $y = \frac{1}{x^n}$ with $n > 0$, the results found in 1-3 above are no longer true, but something similar happens. What can you say about these lengths and areas for $y = \frac{1}{x^n}$?

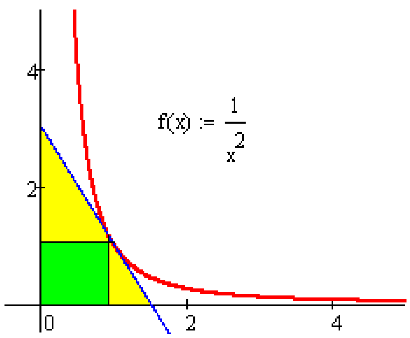
We will try an example, before moving to the general case.

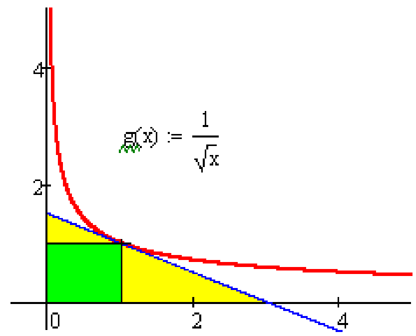
If, $y = \frac{1}{x^2}$, we have $T(x) = \frac{1}{a^2} - \frac{2}{a^3}(x - a) = \frac{-2x}{a^3} + \frac{3}{a^2}$. The coordinates of the point of tangency are $\left(a, \frac{1}{a^2}\right)$. The intercepts (endpoints of the segment in the first quadrant) are $\left(0, \frac{3}{a^2}\right)$ and $\left(\frac{3a}{2}, 0\right)$. The length of this segment is $\sqrt{\left(\frac{3}{a^2} - 0\right)^2 + \left(0 - \frac{3a}{2}\right)^2} = \sqrt{\frac{9}{a^4} + \frac{9a^2}{4}} = \frac{3}{2a^2}\sqrt{4 + a^6}$. The length of a segment to the point of tangency is $\sqrt{(a-0)^2 + \left(\frac{1}{a^2} - \frac{3}{a^2}\right)^2} = \sqrt{a^2 + \frac{4}{a^4}} = \frac{1}{a^2}\sqrt{4 + a^6}$.
We see that the point of tangency is always breaks the segment into 1/3 and 2/3 lengths. Would $y = \frac{1}{x^3}$ break into ¼ and ¾'s?

If $y = \frac{1}{x^n}$, then $y' = \frac{-n}{x^{n+1}}$. So the tangent line is $T(x) = \frac{1}{a^n} - \frac{n}{a^{n+1}}(x - a) = \frac{-nx}{a^{n+1}} + \frac{n+1}{a^n}$. The coordinates of the point of tangency are $\left(a, \frac{1}{a^n}\right)$. The intercepts (endpoints of the segment in the first quadrant) are $\left(0, \frac{n+1}{a^n}\right)$ and $\left(\frac{(n+1)a}{n}, 0\right)$.

The length of this segment is $\sqrt{\left(\frac{n+1}{n}\right)^2 a^2 + \frac{(n+1)^2}{a^n}} = \frac{a(n+1)}{n}\sqrt{1 + \frac{n^2}{a^{n+2}}}$. It does depend upon the point of tangency $x = a$.

The length of the segment to the point of tangency is $\sqrt{a^2+\frac{n^2}{a^n}}=a\sqrt{1+\frac{n^2}{a^{2n+2}}}$, so the ratio of lengths is $\dfrac{a\sqrt{1+\frac{n^2}{a^{2n+2}}}}{\frac{a(n+1)}{n}\sqrt{1+\frac{n^2}{a^{n+2}}}}=\dfrac{n}{n+1}$. This ratio is also an invariant, and does not depend upon the point of tangency. So, yes, $y=\frac{1}{x^3}$ breaks the segment into ¼ and ¾'s while $y=\frac{1}{\sqrt{x}}$ breaks it into 2/3 and 1/3.





The ratio of areas is not quite so nice, but it is also invariant. The coordinates of the endpoints are $\left(0,\frac{n+1}{a^n}\right)$ and $\left(\frac{(n+1)a}{n},0\right)$, so area of the triangle is $\left(\frac{1}{2}\right)\left(\frac{n+1}{a^n}\right)\left(\frac{(n+1)a}{n}\right)=\frac{(n+1)^2}{2n\left(a^{n-1}\right)}$.

The area of the rectangle is $(a)\left(\frac{1}{a^n}\right)=\frac{1}{a^{n-1}}$.

The ratio, then, is $\frac{2n}{(n+1)^2}$ and is again independent of the location of the point of tangency.

The power that maximizes the ratio can be found. If $R(n)=\frac{2n}{(n+1)^2}$, then

$R'(n)=\frac{(n+1)^2 2-(2n)2(n+1)}{(n+1)^4}=\frac{(n+1)2-(4n)}{(n+1)^3}=\frac{-2n+2}{(n+1)^3}$. The critical values are at $n=1,-1$.

We are only interested in $n>0$. $R''(n)=\frac{4n-8}{(n+1)^4}$ and $R''(1)<0$, so $y=\frac{1}{x}$ has the maximum ratio of areas.

5. The portion of the implicit function $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$ in the first quadrant has a similar interesting property. What property or properties can you find? (if you haven't studied implicit differentiation yet, solve for *y*, $y=\left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}$, and treat it like an ordinary function in the 1st quadrant)

This one is messy whether you treat it implicitly or as an explicit function. For the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$, we have $\frac{2}{3}x^{\frac{-1}{3}}+\frac{2}{3}y^{\frac{-1}{3}}\left(\frac{dy}{dx}\right)=0$, so $\left(\frac{dy}{dx}\right)=\frac{-x^{\frac{-1}{3}}}{y^{\frac{-1}{3}}}=-\sqrt[3]{\frac{y}{x}}$.

The point of tangency is $\left(a,\sqrt{\left(1-a^{\frac{2}{3}}\right)^{3}}\right)$ and the equation of the tangent line at *x* = *a* is

$T(x)=-\frac{\sqrt{1-a^{2/3}}x}{a^{1/3}}+\sqrt{1-a^{2/3}}$. The endpoints of the segment are then $\left(0,\sqrt{1-a^{\frac{2}{3}}}\right)$ and $\left(a^{1/3},0\right)$.

The length of this segment is $\sqrt{a^{\frac{2}{3}}+\left(1-a^{\frac{2}{3}}\right)}=1$. Yikes! The length is always 1 no matter what point of tangency we pick.

The length of the segment to the point of tangency $\left(a,\sqrt{\left(1-a^{\frac{2}{3}}\right)^{3}}\right)$ is

$\sqrt{\left(a-a^{\frac{1}{3}}\right)^{2}+\left(1-a^{2/3}\right)^{3}}=\sqrt{a^{4/3}-2a^{2/3}+1}=\sqrt{\left(a^{2/3}-1\right)^{2}}=\left|a^{2/3}-1\right|$. The ratio of lengths does depend upon the location of *a*.

The area of the triangle is $\left(\frac{1}{2}\right)\left(\sqrt[3]{a}\right)\left(\sqrt{1-a^{2/3}}\right)$ and the area of the rectangle is $(a)\left(1-a^{2/3}\right)\sqrt{1-a^{2/3}}=\left(a-a^{5/3}\right)\sqrt{1-a^{2/3}}$. The ratio, then, is $\frac{2\left(a-a^{5/3}\right)}{\left(\sqrt[3]{a}\right)}=2a^{2/3}-2a^{4/3}$ and is dependent on the location of the point of tangency.

Let $R(a)=2a^{2/3}-2a^{4/3}$, so $R'(a)=\frac{4}{3}a^{-1/3}-\frac{8}{3}a^{1/3}$. If $R'(a)=0$, then $a^{-1/3}=2a^{1/3}$. The critical values are *a* = 0 and $a=\pm\frac{1}{2\sqrt{2}}$. We have $R(0)=0$ and $R\left(\frac{1}{2\sqrt{2}}\right)=\frac{1}{2}$, with $R''\left(\frac{1}{2\sqrt{2}}\right)<0$. The maximum ratio is found when $a=\frac{1}{2\sqrt{2}}$ when the rectangle is half the area of the triangle.

So, it looks like the most interesting aspect of this function is the length of the segment in the first quadrant is always 1, regardless of the location of the point of tangency.

So, this must be the kind of path a falling ladder takes as it slides down a wall. I wonder if it is possible to make a calculus problem out of that?