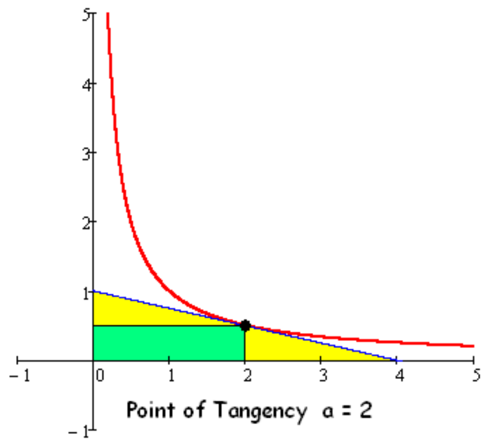
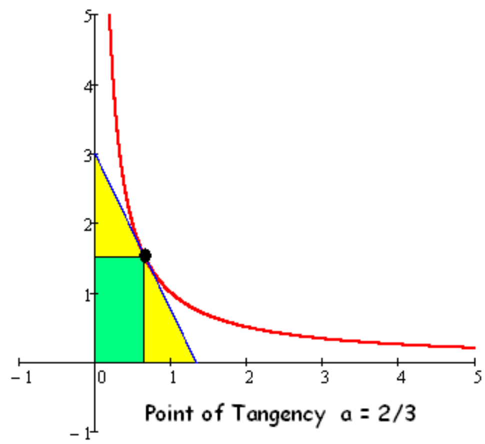
Calculus Challenge #2

Solution due: November 4, 2009

Consider the portion of the graph of $y = \frac{1}{x}$ in the first quadrant.





1. Show that the segment of the tangent to the curve in the first quadrant is bisected at the point of contact. This is true for every point of tangency ($x = a$).

2. How does the area of the triangle bounded by the coordinate axes and the tangent segment vary with the point of tangency?

3. Find the point of tangency, $x = a$, at which the ratio of the area of the rectangle with opposite vertices at (0, 0) and at $\left(a, \frac{1}{a}\right)$ to the area of the triangle defined in 2) is maximum.

4. If we alter the function to $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$, $y = \frac{1}{\sqrt{x}}$, or in general, $y = \frac{1}{x^n}$ with $n > 0$, the results found in 1-3 above are no longer true, but something similar happens. What can you say about these lengths and areas for $y = \frac{1}{x^n}$?

5. The portion of the implicit function $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ in the first quadrant has a similar interesting property. What property or properties can you find? (if you haven't studied implicit differentiation yet, solve for y, $y = \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$, and treat it like an ordinary function in the 1st quadrant)

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