

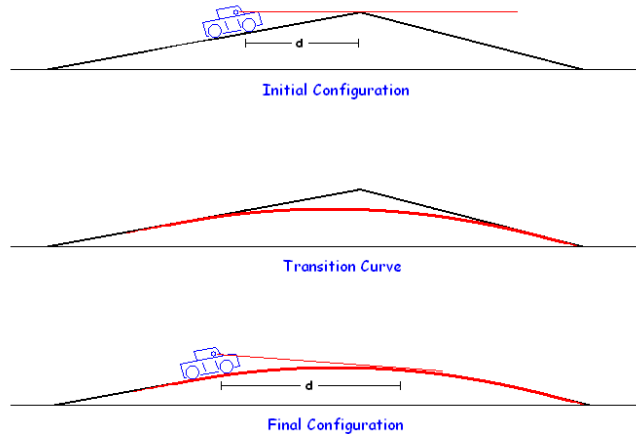
Calculus Challenge #3

SOLUTIONS

In this problem, we are interested in creating a transition curve. A transition curve is a curve in the plan which joins smoothly two line segments having different slopes (typically one positive and the other negative).

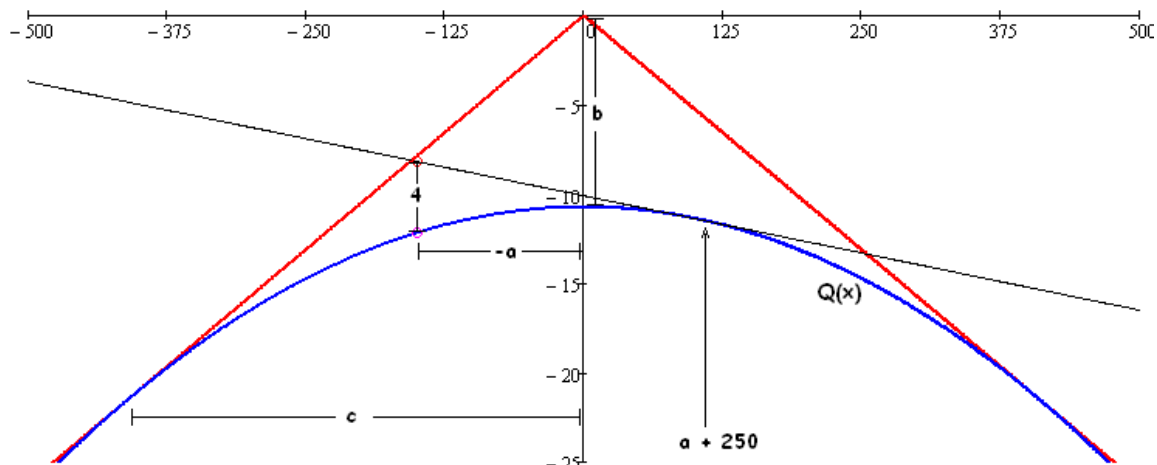
Consider the situation in designing a highway system. The highway proceeds from an uphill segment with slope S_1 to a downhill slope S_2 . If no transition curve is cut, the visibility just before the top of the hill will be dangerously short.

Engineers want to join these two segments with a transition curve so that the horizon as seen from someone in the driver's seat h feet above the road surface is always at least d feet away.



- a) Find a quadratic function that can be used as a transition curve to allow at least 250 feet of visibility down a road which joins an uphill slope of 3 degrees to a downhill slope of 3 degrees. Assume the driver's eye is 4 feet vertically above the road.

The highway slope of 3 degrees can be represented by a line with a slope of $\tan\left(\frac{\pi}{60}\right)$. If we begin by defining our origin to be the intersection of the two linear highway segments, we can write the equations of the lines as $y = \tan\left(\frac{\pi}{60}\right)x$ and $y = -\tan\left(\frac{\pi}{60}\right)x$ and the equation of the transition quadratic as $Q(x) = -kx^2 - b$. We further define the points of tangency to be $(c, Q(c))$ and $(-c, Q(-c))$ as shown in the diagram below.



We seek the values of c , k , and b , so that the tangent to the quadratic at any point $(a, Q(a))$ passes through the point $(a - 250, Q(a - 250) + 4)$.

The tangent at $(a, Q(a))$ has the equation $y = Q(a) + Q'(a)(x - a)$ so

$$y = (-ka^2 - b) - (2ka)(x - a)$$

If this line contains $(a - 250, Q(a - 250) + 4)$, then

$$-k(a - 250)^2 - b + 4 = (-ka^2 - b) - (2ka)((a - 250) - a).$$

Solving for k , we find that $k = \frac{4}{250^2} = \frac{1}{125^2}$. Notice that this value of k does not involve the slope of the lines in question. Its value is determined entirely by the 250 foot view requirement. This will be very important in part e).

We have $Q(x) = -\frac{x^2}{125^2} - b$ as our basic quadratic. We need to solve for b .

We know that at the point of intersection $(c, Q(c))$ the line is tangent to the curve, so

$-\tan\left(\frac{\pi}{60}\right) = -2\left(\frac{c}{125^2}\right)$ which means that $c = \left(\frac{125^2}{2}\right)\tan\left(\frac{\pi}{60}\right) \approx 409.4$. At $x = c$, the curves

$y = -\tan\left(\frac{\pi}{60}\right)x$ and $Q(x) = -\frac{x^2}{125^2} - b$ share the same point, so, $-\frac{c^2}{125^2} - b = -\tan\left(\frac{\pi}{60}\right)c$

and $b = \left(\frac{125^2}{2}\right)\tan^2\left(\frac{\pi}{60}\right) \approx 10.729$.

So, our transition curve is $Q(x) = -\frac{x^2}{125^2} - \left(\frac{125^2}{2}\right)\tan^2\left(\frac{\pi}{60}\right)$ or $Q(x) = -0.000064x^2 - 10.729$

which extends for a length of about 819 linear feet bridging the two linear segments.

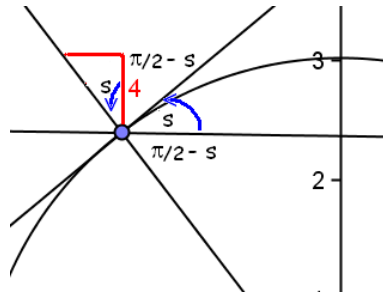
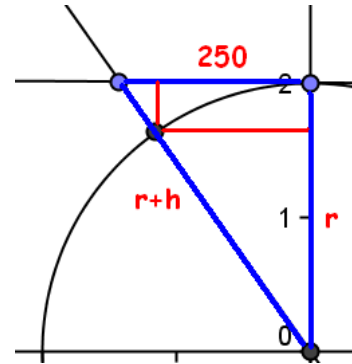
b) Use a circular sector to solve the problem posed in a).

A circle has a constant curvature, and all tangents are perpendicular to the radius at the point of tangency, so we should be able to use that information here.

The problem is easy if we use the perpendicular length of 4 and a horizontal distance from the eye of 250 (in blue). In this case we have $r^2 + 250^2 = (r + h)^2$, so $250^2 = 2rh + h^2$ and

$$r = \frac{250^2 - h^2}{2h}.$$

If we let $h = 4$, we are almost done. But it is not this perpendicular piece that is 4 feet and the horizontal distance should be from the base (in red). How big a difference can this make?



We have a right triangle with an angle s of $\frac{\pi}{60}$ and a leg of 4, so the hypotenuse has a length of $\frac{4}{\cos\left(\frac{\pi}{60}\right)} \approx 4.005$. The horizontal leg has a length of $4 \tan\left(\frac{\pi}{60}\right) \approx 0.21$. The differences between 4 and 4.005 and between 250 and 250.21 are too small to be of any practical importance and a road cannot be built that would distinguish between the two. From a modeling perspective, we can choose either model and engineers will generally choose a worst case and work from there.

Since having 250 feet of vision from 4 feet vertically is equivalent to having 250.21 feet from 4.005 feet perpendicularly, we can use our $r = \frac{250^2 - h^2}{2h}$ formulation with

$$r = \frac{250.21^2 - 4.005^2}{2(4.005)} \approx 7813.23. \text{ We need a circle of radius 7813 to satisfy this criterion. Now,}$$

which circle of this radius? Our circle has the equation, $x^2 + (y + k)^2 = 7813.23^2$.

As before, at the points of tangency, the slope is $\pm \tan\left(\frac{\pi}{60}\right)$. In the first quadrant, for our circle, we know that $2x + 2(y + k)\left(\frac{dy}{dx}\right) = 0$, so $\left(\frac{dy}{dx}\right) = \frac{-x}{(y + k)} = -\tan\left(\frac{\pi}{60}\right)$. At the point

$$\left(c, \sqrt{r^2 - c^2} - k\right), \text{ we have } \left(\frac{dy}{dx}\right) = \frac{-c}{\left(\sqrt{r^2 - c^2}\right)} = -\tan\left(\frac{\pi}{60}\right), \text{ so } c = \frac{r \tan\left(\frac{\pi}{60}\right)}{\sqrt{1 + \tan^2\left(\frac{\pi}{60}\right)}} = r \sin\left(\frac{\pi}{60}\right).$$

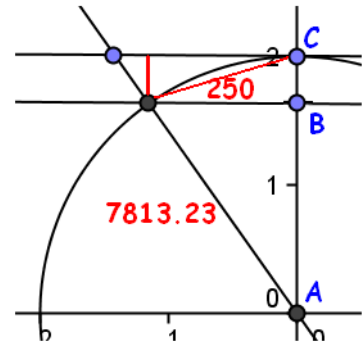
This is evaluated as $(7813.23)\sin\left(\frac{\pi}{60}\right) \approx 408.9$. There are essentially the same points of tangency as for the quadratic in part a).

Now, we only need to find the center $(0, k)$ for our circle $x^2 + (y + k)^2 = 7813.23^2$. Since the curvature of a circle is the same everywhere, we can pick a nice spot to consider this question. Think about the 250 feet being horizontal. Then we have two right triangles.

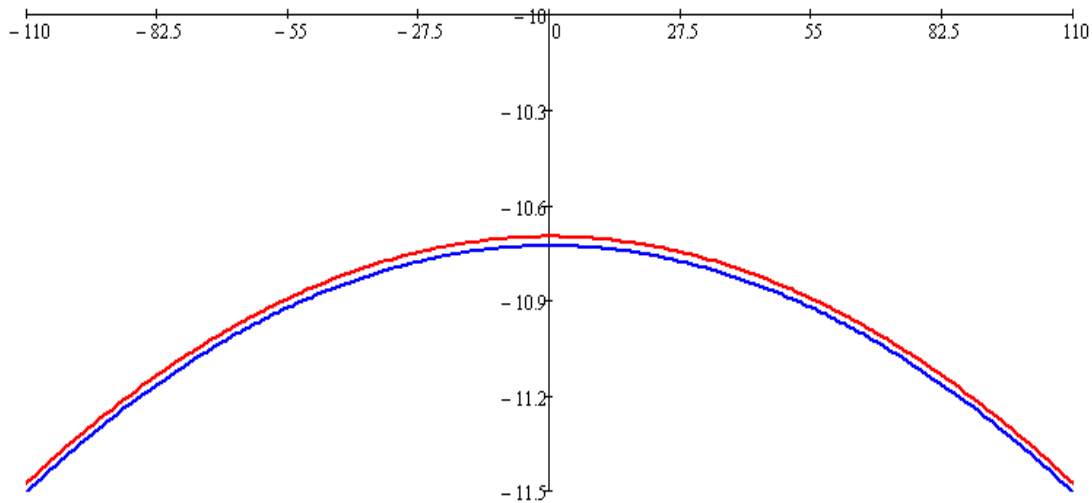
Segment AB has length $\sqrt{7813.23^2 + 250^2} \approx 7817.23$ and segment

BC has length $250 \tan\left(\frac{\pi}{60}\right) \approx 13.12$ for a total length of 7830.33,

So, our circle is $x^2 + (y + 7830.33)^2 = 7813.23^2$.



Compare the two transition curves. They are only a few inches apart at most.



c) Generalize your result. Find a curve that can be used to allow at least d feet of visibility down a road which joins an uphill slope of slope S to a downhill slope of S . Assume the driver's eye is h feet vertically above the road.

We can generalize the quadratic model by repeating the analysis in a) with the parameter values instead of numerical values. So, we want the values of c , k , and b , so that the tangent to the quadratic at any point $(a, Q(a))$ passes through the point $(a - d, Q(a - d) + h)$.

The tangent at $(a, Q(a))$ has the equation $y = Q(a) + Q'(a)(x - a)$ so

$$y = (-ka^2 - b) - (2ka)(x - a).$$

If this line contains $(a-d, Q(a-d)+h)$, then $-k(a-d)^2 - b + h = (-ka^2 - b) - (2ka)((a-d) - a)$. Simplifying, we find that $-k(a^2 - 2da + d^2) + 4 = (-ka^2) + (2dka)$ so, $-k(d^2) + h = 0$ and $k = \frac{h}{d^2}$. Notice that this value of k does not involve the slope of the lines in question. Its value is determined entirely by the 250 foot view requirement.

We have $Q(x) = -\frac{hx^2}{d^2} - b$.

We know that at the point of intersection $(c, Q(c))$ the line is tangent to the curve, so

$-\tan\left(\frac{S\pi}{180}\right) = -2\left(\frac{hc}{d^2}\right)$ which means that $c = \left(\frac{d^2}{2h}\right)\tan\left(\frac{S\pi}{180}\right)$. At $x = c$, the curves

$y = -\tan\left(\frac{S\pi}{180}\right)x$ and $Q(x) = -\frac{hx^2}{d^2} - b$ share the same point, so, $-\frac{hc^2}{d^2} - b = -\tan\left(\frac{S\pi}{180}\right)c$

and $b = \left(\frac{d^2}{2h}\right)\tan^2\left(\frac{S\pi}{180}\right)$.

So, our transition curve is $Q(x) = -\frac{hx^2}{d^2} - \left(\frac{d^2}{2h}\right)\tan^2\left(\frac{\pi}{60}\right)$.

d) In a) we assumed the driver's eye was 4 feet vertically above the road. A more realistic model would have the driver's eye 4 feet along a perpendicular to the road. Use any method (quadratic, circle, or other function) to find a curve that can be used as a transition curve to allow at least 250 feet of visibility down a road which joins an uphill slope of 3 degrees to a downhill slope of 3 degrees. How much does this distinction affect the solution?

We have already seen that there is no practical difference in considering whether the driver is sitting upright (vertical) or perpendicular to the road. We don't really have to worry about the angle of the driver's seat. Underinflated tires would have a similar effect. These small differences should be ignored.

If we repeat the analysis in b) using a triangle with a hypotenuse of 4, we have

$r = \frac{250.21^2 - 4^2}{2(4)} \approx 7823.23$. Our radius is 10 feet larger, so our circle needs to be move down an

additional 10 feet, so $x^2 + (y + 7840.33)^2 = 7823.23^2$ will do, but, as previously stated, there is essentially no difference in this path and the previous one.

e) Find a curve that can be used to allow at least 250 feet of visibility down a road which joins an uphill slope of 2 degrees to a downhill slope of 3 degrees. Assume, as in a), the driver's eye is 4 feet vertically above the road.

We have seen that the coefficient of the quadratic did not depend on the angles of the roads, so $Q(x) = -\frac{(x-p)^2}{125^2} - b$ will work. All we need to do is find where to situate the curve by finding the values of p and b . After many, many trials, I decided to change coordinate systems so the function $Q^*(x) = -\frac{(x)^2}{125^2}$ is simpler and the lines more complicated. This makes the problem much much easier. We can solve the problem in this coordinate system and substitute back into the original.

At $x = c_1$ the tangent to Q^* has a slope of $\tan\left(\frac{\pi}{90}\right)$, so we want $\tan\left(\frac{\pi}{90}\right) = \frac{-2c_1}{125^2}$. Thus,

$c_1 = \frac{125^2}{-2} \tan\left(\frac{\pi}{90}\right) \approx -272.82$. $Q^*(c_1) \approx -4.76$. Similarly, At $x = c_2$, the tangent to Q^* has a

slope of $-\tan\left(\frac{\pi}{60}\right)$, so we want $-\tan\left(\frac{\pi}{60}\right) = \frac{-2c_2}{125^2}$. Thus, $c_2 = \frac{125^2}{2} \tan\left(\frac{\pi}{60}\right) \approx 409.44$.

$Q^*(c_2) \approx -10.73$. The two tangent lines are $L(x) = -4.76 + \tan\left(\frac{\pi}{90}\right)(x + 272.82)$ and

$R(x) = -10.73 - \tan\left(\frac{\pi}{60}\right)(x - 409.44)$. These two curves intersect at the point (68.29, 7.15), so

our solution, which has this point as the origin, is $Q(x) = -\frac{(x+68.29)^2}{125^2} - 7.15$ as shown below.

