Calculus Challenge Problem #4

Due December 9, 2009

Elasticity of Demand

The economy is much in the news. This bi-week's problem will illustrate a basic principle of supply and demand, and how sensitive revenue is to the type of product being sold. There is a bit of background reading to be done before we can get to the calculus. Due to the Thanksgiving break, we will have three weeks for this problem.

We start by modeling the market for a product with a demand function $D(p)$ that gives the amount of the product that can be sold over a certain time if the product is priced at p dollars per unit. For our study, we assume $D(p)$ is a differentiable function whose graph is decreasing over the domain of reality.

When a manufacturer raises the price of a product, the demand goes down, but what happens to the manufacturer's revenue? Assuming that all of the product in demand is actually sold, revenue is given by

$$R(p) = p \cdot D(p).$$

A price increase can lead to an increase in revenue; however, a price increase can also cause revenue to decrease. For instance, suppose the demand for a particular product is 1000 units when the price is $10 per unit. Since revenue is equal to price times demand, the revenue associated with a price of $10 is $10,000. What will happen if the price increases to $11 per unit? Again, in this case, the price increase will lead to a decrease in demand. If demand falls to 990 units, then revenue will be given by $11 per unit times 990 units, so revenue increases to $10,890. If the demand falls to 900 units, then revenue will be given by $11 per unit times 900 units, so revenue decreases to $9,900. Thus, an increase in price to $11 may lead to either an increase or a decrease in revenue. Whether revenue rises or falls when prices go up depends on the extent to which demand falls in response to the price increase.

Economists have quantified the responsiveness of the demand function to increases in price by comparing changes in the demand to changes in the price. Changes in demand and price are more meaningful when we also know the current demand and price level; for example, a change of $0.10 in the price for an item means one thing if the current price is $10.00 and means something very different if the current price is $0.20. For this reason, economists compare relative changes rather than absolute changes. The relative change in demand is the ratio $\frac{\Delta D}{D}$ and the relative change in price is $\frac{\Delta p}{p}$. If we now look at the ratio of these two relative changes,

$$\frac{\text{relative change in demand } D(p)}{\text{relative change in price } p},$$

we get the expression

$$\frac{\Delta D}{D(p)} \Big/ \frac{\Delta p}{p}.$$

The expression given in the equation above is equivalent to $\frac{\Delta D \cdot p}{\Delta p \cdot D(p)}$, which we can separate into a product of two ratios as

$$\left(\frac{\Delta D}{\Delta p} \cdot\right)\frac{p}{D(p)}$$

This quantity gives the ***elasticity of demand*** over a price interval from p to $p + \Delta p$. The function that gives the elasticity of demand for each point on the demand curve is defined as the limit of the elasticity of demand over an interval as the length of the interval Δp goes to zero. In symbols, elasticity of demand is given by

$$E(p) = \lim_{\Delta p \to 0} \frac{\Delta D}{\Delta p} \cdot \frac{p}{D(p)},$$

which, by the definition of the derivative, is equivalent to

$$E(p) = \left(\frac{dD}{dp} \cdot\right)\frac{p}{D(p)}.$$

To see how the elasticity of demand relates to revenue we can look at the rate of change of the revenue function *R*.

(1) Find $\frac{dR}{dp}$, the rate of change of revenue with respect to price, by differentiating $R(p) = p \cdot D(p)$. Show that $\frac{dR}{dp} = D(p)[1 + E(p)]$ after some simplification.

(2) There are three possibilities for the behavior of revenues at a price $p = p^*$ that follow from the equation $\frac{dR}{dp}$ in terms of *D* and *E*. (We are assuming that $D(p)$ has a negative slope, so demand decreases as price increases.)

a. When $-1 < E(p^*) < 0$, demand is said to be ***inelastic***. Describe what you know about *R'* and *R* in this case and what this means in context.
b. When $E(p) < -1$, demand is said to be ***elastic***. Describe what you know about *R'* and *R* in this case and what this means in context.
c. When $E(p) = -1$, demand is said to be ***unit elastic***. Describe what you know about *R'* and *R* in this case and what this means in context.

The elasticity of demand for a particular product is a relative measure of responsiveness and is thus independent of market size, so we can compare the elasticity of different products. The demand for necessities like food and clothing is usually inelastic, while the demand for discretionary items is often elastic.

(3) Suppose the demand function for a particular graphing calculator is

$$D(p) = 1000 \cdot e^{-0.015p}$$

a. What is the elasticity of demand associated with a price of $100?
b. What is the elasticity of demand associated with a price of $60?
c. Graph *D(p), R(p), E(p),* and *R' (p).* Explain the relationships between the graphs of *D(p), R(p), E(p),* and *R'* (*p)*
d. Investigate the change in revenue associated with each price to determine how the elasticity of demand is related to the way that revenue responds to price increases.

(4) Terrence J. Wales described the demand for distilled spirits with the equation $q = -0.00375p + 7.87$ with *p* representing the retail price of liquor in dollars per case and *q* representing the average number of cases of distilled spirits purchased annually by individual consumers.

a. What is the function of elasticity in terms of price?
b. What is the elasticity if liquor costs $52.80 per case? What does this value mean?

(5) Nisbet and Vakil modeled the market for marijuana with the demand equation

$$D(p) = 15.4p^{-1.013}.$$

If the goal of the police is to decrease the revenue for drug dealers, law enforcement can either focus their attention on buyers (which decreases demand) or they can focus on dealers (which increases prices). Use the concept of elasticity to investigate how revenue would change as a result of focusing on dealers versus focusing on buyers.

(6) What demand function has an elasticity of -1 for all values of *p?*

References:

Bartkovich, Kevin, et al, *Contemporary Calculus through Applications*, 166-170, 1996.

Wales, Terrence J.; “Distilled Spirits and Interstate Consumption Effects,” *The American Economic Review* 57(4), 1968.

Nisbet, C. T. and F. Vakil, “Some Estimates of Price and Expenditure Elasticities of Demand for Marijuana among UCLA Students.” *The Review of Economics and Statistics* 54(4), 1972.