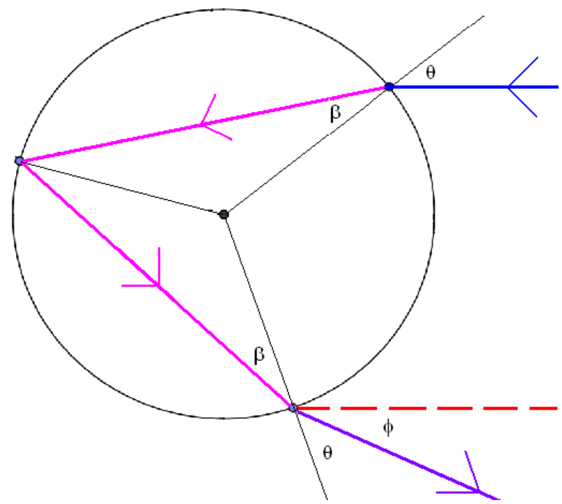
Calculus Challenge #5

Due January, 6, 2010

The diagram below shows the path of a ray of light entering a spherical raindrop an leaving again, after undergoing one internal reflection. The change of direction at the air-water interface is governed by Snell's Law of Refraction. According to Snell's Law, $\sin(\beta) = k\sin(\theta)$ where *k* is a positive constant known as the refractive index. In this example, we will use *k* = 0.75.

Angles β and θ are measured with respect to lines that are perpendicular to the air-water interface (perpendicular to the tangent to the circle).

The latitude of the incoming ray (blue) is θ and the angle of depression of the outgoing ray is ϕ. The red dotted line in the diagram is parallel to the incoming ray.



1. Show that $\beta = 0.5681$ when $\theta = 0.8$ (radians).

This is basic trigonometry. We are given that $\sin(\beta) = 0.75\sin(0.8)$, so $\sin(\beta) = 0.53801$ and $\beta = \sin^{-1}(0.53801)$. So, $\beta = 0.5681$.

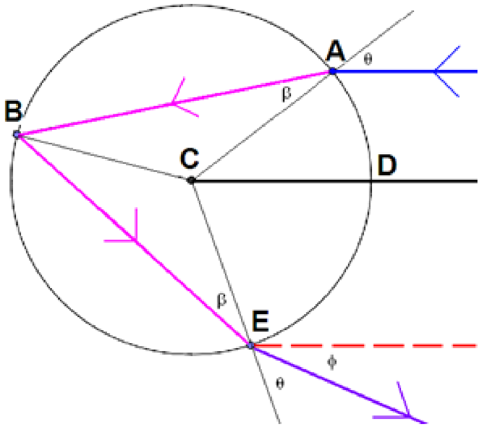
2. It can be shown that $\phi = 4\beta - 2\theta$. Use this equation and Snell's Law to express ϕ as a function of θ. Show that $\phi = 0.6723$ when $\theta = 0.8$. (as an ungraded, but interesting, component to this challenge, prove $\phi = 4\beta - 2\theta$)

We are given $\phi = 4\beta - 2\theta$ and $\beta = \sin^{-1}(0.53801)$. So, by substitution we have an equation for ϕ in terms of θ.

$$\phi = 4\sin^{-1}(0.75\sin(\theta)) - 2\theta$$

When $\theta = 0.8$, this is $\phi = 4\sin^{-1}(0.75\sin(0.8)) - 2(0.8)$, so $\phi = 4(0.5681) - 1.6$ and $\phi = 0.6723$.

To see that $\phi = 4\beta - 2\theta$, we need to do some geometry. Actually, Pine View School's solution came in early and was so much nicer than the convoluted way I used to figure it out, I'm using their diagram and argument. Thanks guys!



Let the points be labeled as they are in the diagram on the right, and let CD be horizontal. If two parallel lines are cut by a transversal, the corresponding angles are congruent, so $\angle ACD = \theta$ and $\angle DCE = \phi + \theta$. Therefore, $\angle ACE = 2\theta + \phi$. Because an intercepted arc is equal to the measure of the central angle, minor arc $AE = 2\theta + \phi$. All radii of a circle are congruent, so $AC = BC$ and $BC = CE$. Therefore, triangles ABC and BCE are isosceles, so $\angle ABC = \angle BAC = \beta$ and $\angle CBE = \angle CEB = \beta$. By addition, $\angle ABE = 2\beta$. Because an angle inscribed in a circle is half the measure of the intercepting arc, arc AE is equal to twice the measure of $\angle ABE$, so $AE = 4\beta$. By transitivity, $4\beta = 2\theta + \phi$, so $\phi = 4\beta - 2\theta$.

3. As θ increases from 0 to $\frac{\pi}{2}$, angle ϕ increases from 0 to a maximum value and then decreases. Find $\frac{d\phi}{d\theta}$ and use it to find the maximum value of ϕ. This angle determines where a rainbow appears in the sky after a late-afternoon thunderstorm.

We have found that $\phi = 4\sin^{-1}(0.75\sin(\theta)) - 2\theta$, so we do the classic max-min thing.

$$\frac{d\phi}{d\theta} = 4\left(\frac{0.75\cos(\theta)}{\sqrt{1-(0.75\sin(\theta))^2}}\right) - 2 = \left(\frac{12\cos(\theta)}{\sqrt{16-9\sin^2(\theta)}}\right) - 2.$$ We need to solve $\frac{d\phi}{d\theta} = 0$ numerically. We find that $\theta \approx 1.03657$.

If $\theta \approx 1.03657$, then $\phi = 4\sin^{-1}(0.75\sin(1.03657)) - 2(1.03657)$. $\phi = 0.73356$.
The maximum angle is $\phi = 0.7336$ radians.

4. The maximum value of the rainbow angle ϕ depends on the refractive index *k*. The value of *k* depends on the color of the incident light. The index *k* = 0.75 belongs to yellow light. The indices for the extreme colors of the spectrum are *k* = 0.7513 for red and *k* = 0.7435 for violet. For each, find the corresponding maximum value of ϕ. Then show that the apparent width of a rainbow is about 2 degrees (about 4 times the apparent diameter of the moon).

We repeat the computations in part 3, using *k* = 0.7513 for red and *k* = 0.7435 for violet, which represent the extremes.

For red light, which has k = 0.7513, and $\phi = 4\sin^{-1}(0.7513\sin(\theta)) - 2\theta$. So,

$$\frac{d\phi}{d\theta} = 4\left(\frac{0.7513\cos(\theta)}{\sqrt{1-(0.7513\sin(\theta))^2}}\right) - 2.$$ In the interval $0 \le \theta \le \frac{\pi}{2}$, $\frac{d\phi}{d\theta} = 0$ only when $\theta = 1.03891$.

If $\theta = 1.03891$, we have $\phi = 4\sin^{-1}(0.7513\sin(1.03891)) - 2(1.03891)$ so $\phi = 0.73943$.

The maximum value of ϕ is 0.7394 radians.

For violet light, which has k = 0.7435, $\phi = 4\sin^{-1}(0.7435\sin(\theta)) - 2\theta$

$$\frac{d\phi}{d\theta} = 4\left(\frac{0.7435\cos(\theta)}{\sqrt{1-(0.7435\sin(\theta))^2}}\right) - 2.$$ As before, $\frac{d\phi}{d\theta} = 0$ when $\theta = 1.02477$

When $\theta = 1.02477$, $\phi = 4\sin^{-1}(0.7435\sin(1.02477)) - 2(1.02477)$. $\phi = 0.70452$.

So ϕ has a maximum value of 0.7045 radians.

The two extremes red and violet differ by about $0.73943 - 0.70452 = 0.03491$ radians. This is 2.0002 degrees which is the apparent width of a rainbow.

Reference: Philips Exeter Academy Mathematics 4C