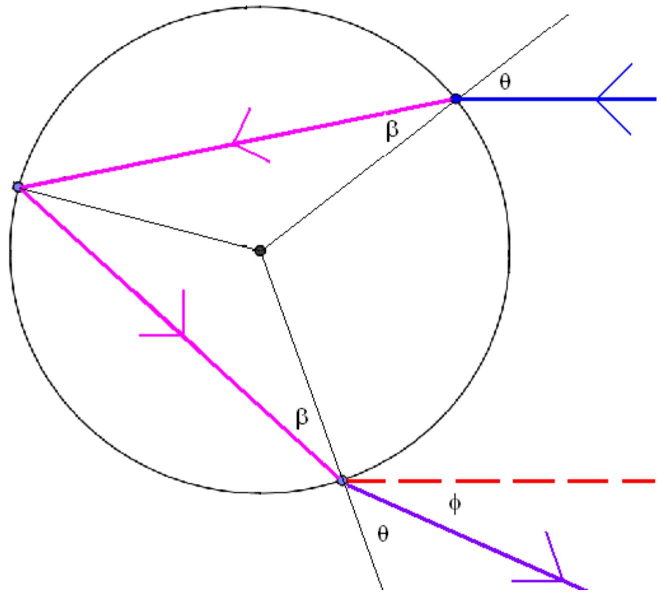
Calculus Challenge #5

Due January, 6, 2010

The diagram below shows the path of a ray of light entering a spherical raindrop an leaving again, after undergoing one internal reflection. The change of direction at the air-water interface is governed by Snell's Law of Refraction. According to Snell's Law, $\sin(\beta) = k\sin(\theta)$ where k is a positive constant known as the refractive index. In this example, we will use k = 0.75.

Angles β and θ are measured with respect to lines that are perpendicular to the air-water interface (perpendicular to the tangent to the circle).



The latitude of the incoming ray (blue) is θ and the angle of depression of the outgoing ray is ϕ. The red dotted line in the diagram is parallel to the incoming ray.

1. Show that $\beta = 0.5681$ when $\theta = 0.8$ (radians).

2. It can be shown that $\phi = 4\beta - 2\theta$. Use this equation and Snell's Law to express ϕ as a function of θ. Show that $\phi = 0.6723$ when $\theta = 0.8$.
(as an ungraded, but interesting, component to this challenge, prove $\phi = 4\beta - 2\theta$)

3. As θ increases from 0 to $\frac{\pi}{2}$, angle ϕ increases from 0 to a maximum value and then decreases. Find $\frac{d\phi}{d\theta}$ and use it to find the maximum value of ϕ. This angle determines where a rainbow appears in the sky after a late-afternoon thunderstorm.

4. The maximum value of the rainbow angle ϕ depends on the refractive index k. The value of k depends on the color of the incident light. The index k = 0.75 belongs to yellow light. The indices for the extreme colors of the spectrum are k = 0.7513 for red and k = 0.7435 for violet. For each, find the corresponding maximum value of ϕ. Then show that the apparent width of a rainbow is about 2 degrees (about 4 times the apparent diameter of the moon).

Reference: Philips Exeter Academy Mathematics 4C