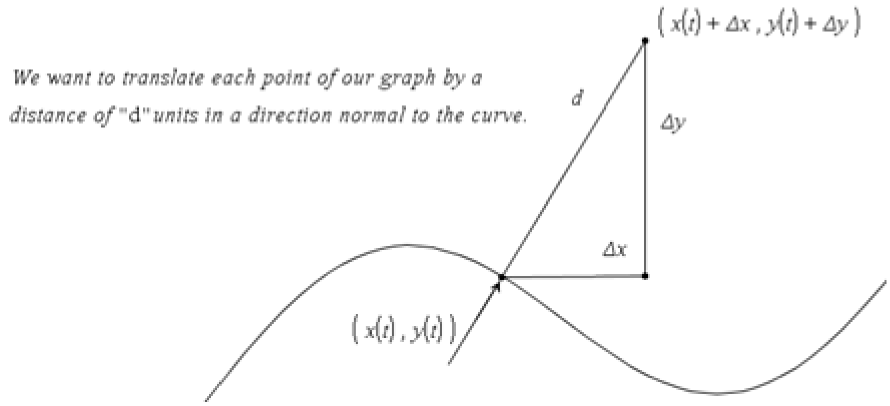
Calculus Challenge #6 Solution

Given a curve defined parametrically by $x = f(t)$ and $y = g(t)$, find the equations of the curve that is always *d* units away.



A curve (**O** for original) defined by the parametric equations $x = f(t)$, $y = g(t)$ has a "parallel" curve a perpendicular distance *d* away. The new curve (**N**) can be defined parametrically by $x = f^*(t)$ and $y = g^*(t)$.

The slope of the line tangent to **O** is always $\left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right)$, so a segment normal to **O** has a slope of $-\left(\frac{dx}{dt}\right) \Big/ \left(\frac{dy}{dt}\right)$ or $\dfrac{-f'(t)}{g'(t)}$.

The equation of a line perpendicular to **O** is $y - g(t) = \left(\dfrac{-f'(t)}{g'(t)}\right)\left(x - f(t)\right)$ and

$\dfrac{y - g(t)}{x - f(t)} = \left(\dfrac{-f'(t)}{g'(t)}\right) = \dfrac{g^*(t) - g(t)}{f^*(t) - f(t)}$ since **N** is parallel to **O** (at least locally).

This means $\left(g^*(t) - g(t)\right) = \left(\dfrac{-f'(t)}{g'(t)}\right)\left(f^*(t) - f(t)\right)$. From the right triangle, we also have $\left(g^*(t) - g(t)\right)^2 + \left(f^*(t) - f(t)\right)^2 = d^2$.

Putting this together, we find that $\left[\left(\dfrac{-f'(t)}{g'(t)}\right)\left(f^*(t) - f(t)\right)\right]^2 + \left(f^*(t) - f(t)\right)^2 = d^2$ so

$\left[\left(\dfrac{-f'(t)}{g'(t)}\right)^2 + 1\right]\left(f^*(t) - f(t)\right)^2 = d^2$.

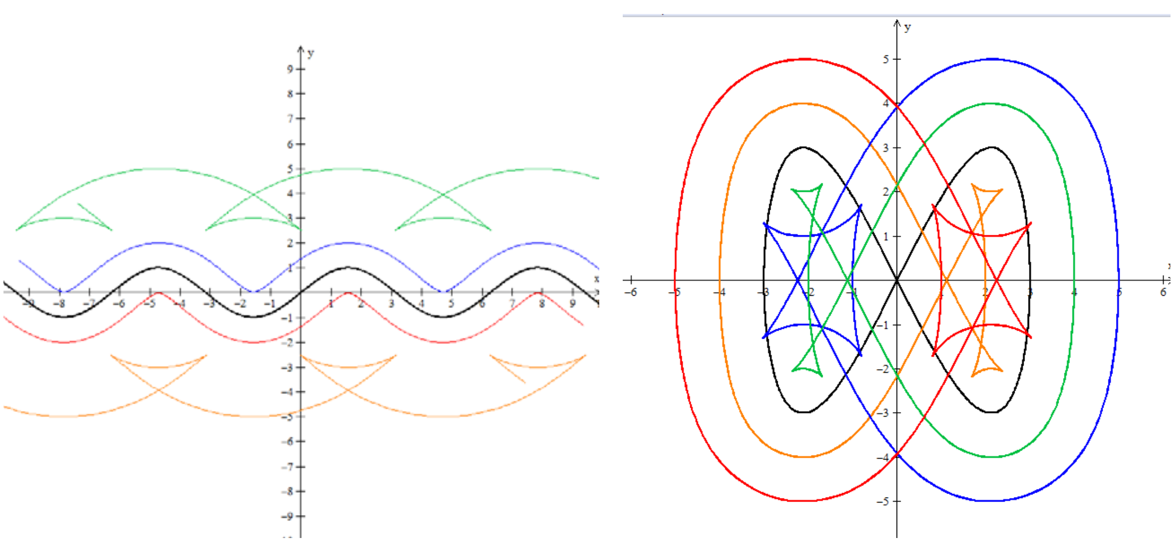
Solving for $f^*(t)$ we find $\left|f^*(t)-f(t)\right|=\dfrac{|d|}{\sqrt{\left(\dfrac{-f'(t)}{g'(t)}\right)^2+1}}=\dfrac{d\,|g'(t)|}{\sqrt{(f'(t))^2+(g'(t))^2}}$ (*d* is positive). If **N** is to the right of **O**, then $f^*(t)-f(t)>0$ and $f^*(t)-f(t)=\dfrac{d\,|g'(t)|}{\sqrt{(f'(t))^2+(g'(t))^2}}$, so $f^*(t)=f(t)+\dfrac{d\,|g'(t)|}{\sqrt{(f'(t))^2+(g'(t))^2}}$.

If **N** is to the left of **O**, then $f^*(t)-f(t)<0$ and $-f^*(t)+f(t)=\dfrac{d\,|g'(t)|}{\sqrt{(f'(t))^2+(g'(t))^2}}$, so

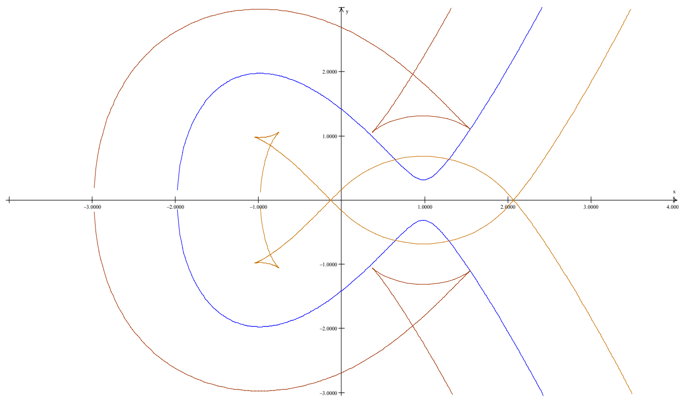
$$f^*(t)=f(t)-\frac{d\,|g'(t)|}{\sqrt{(f'(t))^2+(g'(t))^2}}.$$

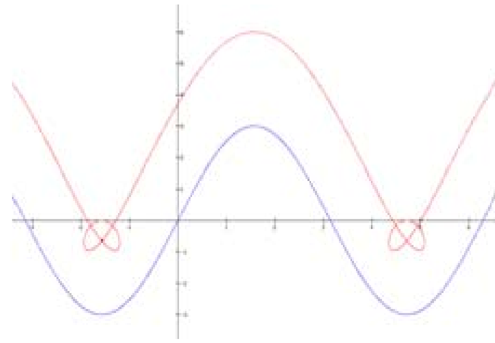
In a similar fashion, we find that $g^*(t)=g(t)\pm\dfrac{d\,|f'(t)|}{\sqrt{(f'(t))^2+(g'(t))^2}}$.

Everyone noticed that curves that are too tight (too large a curvature) create problems with cusps and that the points on the "parallel" curves **N** can be closer than *d* units from some point on the original **O**. Here are some of the examples submitted.



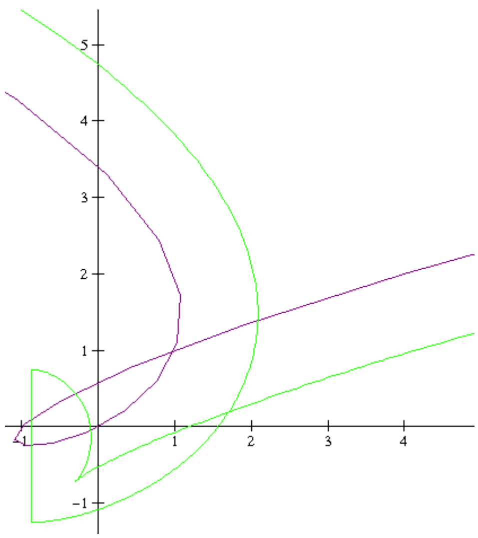
From Benjamin N. Cardozo High School





From Pine View School

From Hickman High School



From Glenn Eagle High School