Calculus Challenge Problem #7

Solution

Non-Fundamental Functions: When Wrong is Right!

A student was wondering why I took off points for his solution to the following integration problem.

$$\int_0^1 x^2 + 2x + \tfrac{5}{3}\, dx = \left(1^2\right) + 2\left(1\right) + \tfrac{5}{3} - \left(\left(0^2\right) + 2\left(0\right) + \tfrac{5}{3}\right) = 3.$$

First, note the error in the student's work. I patiently showed him that the correct method was

$$\int_0^1 x^2 + 2x + \tfrac{5}{3}\, dx = \left(\frac{x^3}{3} + x^2 + \tfrac{5}{3}x\right)\Big|_0^1 = \left(\frac{1^3}{3}\right) + \left(1^2\right) + \tfrac{5}{3}\left(1\right) - \left(\left(\frac{0^3}{3}\right) + \left(0^2\right) + \tfrac{5}{3}\left(0\right)\right) = 3.$$

He, of course, argued that since his way was easier and got the same answer, it was a better solution. This got me wondering…

Are there other functions for which $\int_0^1 f\left(x\right) dx = f\left(1\right) - f\left(0\right)$? Since the fundamental theorem of calculus is usually required to evaluate a definite integral, we will call a function for which $\int_a^b f\left(x\right) dx = f\left(b\right) - f\left(a\right)$ a *non-fundamental* function on [*a*, *b*]. We see that $f\left(x\right) = x^2 + 2x + \tfrac{5}{3}$ is non-fundamental on [0, 1].

1. Find a linear function that is non-fundamental on [0, 1].

We need to find a function $f\left(x\right) = ax + b$ so that $\int_0^1 ax + b\, dx = \left(a + b\right) - b = a$ and $\int_0^1 ax + b\, dx = \tfrac{1}{2}a + b$, so $a = 2b$. Any function, ($y = 6x + 3,\ y = 2\pi x + \pi,$ etc) will work. The basic function is $f\left(x\right) = kx + \dfrac{k}{2}$.

2. Are there any other quadratic functions that are non-fundamental on [0, 1]?

Sure, lots of them. Now we need that $\int_0^1 ax^2+bx+c\,dx=(a+b+c)-c=a+b$ and $\int_0^1 ax^2+bx+c\,dx=\frac{1}{3}a+\frac{1}{2}b+c$, so $\frac{1}{3}a+\frac{1}{2}b+c=a+b$. So, as long as $c=\frac{2}{3}a+\frac{1}{2}b$, the quadratic will be non-fundamental. The simplest solution is to set b = 0, so $f(x)=kx^2+\frac{2k}{3}$

3. Can you find all polynomials that are non-fundamental on [0, 1]?

Generalizing the previous result, we see that there is always a condition on the constant that will make the sum work. For example, with the 6th degree polynomial $f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6$, we require $a_0=\frac{a_1}{2}+\frac{2a_2}{3}+\frac{3a_3}{4}+\frac{4a_4}{5}+\frac{5a_5}{6}+\frac{6a_6}{7}$. Clearly, we could just as easily choose any other coefficient instead of the constant. Again, the simplest function satisfying these conditions is $f(x)=kx^6+\frac{6k}{7}$. In general, the n^{th} degree polynomial $f(x)=kx^n+k\left(\frac{n}{n+1}\right)$ satisfies the conditions.

4. Must the interval of integration be [0, 1] for polynomials to be non-fundamental? Can we find a more general result?

No, [0, 1] is convenient, but any other interval will do. Consider $f(x)=ax^n+b$ on the interval [L, U]. Then $\int_L^U ax^n+b\,dx=\frac{a}{n+1}\left(U^{n+1}-L^{n+1}\right)+b(U-L)$ must equal $a\left(U^n-L^n\right)$. Solving $\frac{a}{n+1}\left(U^{n+1}-L^{n+1}\right)+b(U-L)=a\left(U^n-L^n\right)$, so $b=a\frac{\left(U^n-L^n\right)}{(U-L)}-\frac{a}{n+1}\frac{\left(U^{n+1}-L^{n+1}\right)}{(U-L)}=a\frac{\left(U^n-L^n\right)}{(U-L)}-\frac{a}{n+1}\frac{\left(U^{n+1}-L^{n+1}\right)}{(U-L)}$. For every a, U, and L, there is a b that works.

5. Can functions other than polynomials be non-fundamental on some interval? If so, give some examples.

The obvious choices are exponentials and sines and cosines. Karl Gross, o Bellville High School contributed the following example:

Consider $f(x)=\ln(1+x)+c$ on [0, 1]. Then $2\ln 2-1+c=\ln 2-\ln 1$, so $c=1-\ln 2=\ln\left(\frac{e}{2}\right)$.

So, $f(x)=\ln(1+x)+\ln\left(\frac{e}{2}\right)$ is non-fundamental on [0, 1]. But we know that the Maclaurin expansion for $f(x) = \ln(1+x)$ is $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \cdots$ with $a_0 = 0$, $a_1 = 1$, $a_2 = \frac{-1}{2}$, $a_3 = \frac{1}{3}$, ….

We have from our earlier work that $a_0 = \frac{a_1}{2} + \frac{2a_2}{3} + \frac{3a_3}{4} + \frac{4a_4}{5} + \frac{5a_5}{6} + \frac{6a_6}{7}$ makes the function non-fundamental, so we must have

$$\ln\left(\frac{e}{2}\right) = \frac{1}{2}(1) + \frac{2}{3}\left(-\frac{1}{2}\right) + \frac{3}{4}\left(\frac{1}{3}\right) + \cdots + \left(\frac{n}{n+1}\right)\left(\frac{(-1)^{n+1}}{n}\right) + \cdots$$

and we can use the non-fundamental property to find the sum of the infinite series.

Reference: Graham, Jeffery A., *Self-Integrating Polynomials*, **The College Mathematics Journal**, September, 2005 as presented in **The Calculus Collection** edited by Caren Diefenderfer and Roger Nelson, MAA.