Non-Fundamental Functions: When Wrong is Right!

A student was wondering why I took off points for his solution to the following integration problem.

\[
\int_{0}^{1} x^2 + 2x + \frac{5}{3} \, dx = \left( 1^2 \right) + 2 \left( 1 \right) + \frac{5}{3} - \left( \left( 0^2 \right) + 2 \left( 0 \right) + \frac{5}{3} \right) = 3.
\]

First, note the error in the student’s work. I patiently showed him that the correct method was

\[
\int_{0}^{1} x^2 + 2x + \frac{5}{3} \, dx = \left[ \frac{x^3}{3} + x^2 + \frac{5}{3}x \right]_{0}^{1} = \left( \frac{1^3}{3} \right) + \left( 1^2 \right) + \frac{5}{3} \left( 1 \right) - \left( \frac{0^3}{3} \right) + \left( 0^2 \right) + \frac{5}{3} \left( 0 \right) = 3.
\]

He, of course, argued that since his way was easier and got the same answer, it was a better solution. This got me wondering…

Are there other functions for which \( \int_{a}^{b} f(x) \, dx = f(b) - f(a) \)? Since the fundamental theorem of calculus is usually required to evaluate a definite integral, we will call a function for which \( \int_{a}^{b} f(x) \, dx = f(b) - f(a) \) a non-fundamental function on \([a, b]\). We see that \( f(x) = x^2 + 2x + \frac{5}{3} \) is non-fundamental on \([0, 1]\).

1. Find a linear function that is non-fundamental on \([0, 1]\).

We need to find a function \( f(x) = ax + b \) so that \( \int_{0}^{1} (ax + b) \, dx = (a + b) - b = a \) and

\[
\int_{0}^{1} (ax + b) \, dx = \frac{1}{2}a + b, \text{ so } a = 2b. \]

Any function, \( (y = 6x + 3, y = 2\pi x + \pi, \text{ etc}) \) will work. The basic function is \( f(x) = kx + \frac{k}{2} \).

2. Are there any other quadratic functions that are non-fundamental on \([0, 1]\)?
Sure, lots of them. Now we need that
\[ \int_0^1 ax^2 + bx + c \, dx = (a + b + c) - c = a + b \]
and
\[ \int_0^1 ax^2 + bx + c \, dx = \frac{1}{2}a + \frac{1}{2}b + c, \]
so \( \frac{1}{2}a + \frac{1}{2}b + c = a + b \). So, as long as \( c = \frac{1}{2}a + \frac{1}{2}b \), the quadratic will be non-fundamental. The simplest solution is to set \( b = 0 \), so \( f(x) = kx^2 + \frac{2k}{3} \).

3. Can you find all polynomials that are non-fundamental on \([0, 1]\)?

Generalizing the previous result, we see that there is always a condition on the constant that will make the sum work. For example, with the 6th degree polynomial
\[ f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6, \]
we require
\[ a_0 = \frac{a_1}{2} + \frac{2a_2}{3} + \frac{3a_3}{4} + \frac{4a_4}{5} + \frac{5a_5}{6} + \frac{6a_6}{7}. \]
Clearly, we could just as easily choose any other coefficient instead of the constant. Again, the simplest function satisfying these conditions is \( f(x) = kx^6 + \frac{6k}{7} \). In general, the \( n \)th degree polynomial \( f(x) = kx^n + k\left(\frac{n}{n+1}\right) \) satisfies the conditions.

4. Must the interval of integration be \([0, 1]\) for polynomials to be non-fundamental? Can we find a more general result?

No, \([0, 1]\) is convenient, but any other interval will do. Consider \( f(x) = ax^n + b \) on the interval \([L, U]\). Then
\[ \int_L^U ax^n + b \, dx = a \left( \frac{U^{n+1} - L^{n+1}}{n+1} \right) + b(U - L) \]
must equal \( a(U^n - L^n) \). Solving
\[ \frac{a}{n+1}(U^{n+1} - L^{n+1}) + b(U - L) = a(U^n - L^n), \]
so
\[ b = a \left( \frac{U^n - L^n}{U - L} \right) - \frac{a}{n+1} \left( \frac{U^{n+1} - L^{n+1}}{U - L} \right) = a \left( \frac{U^n - L^n}{U - L} \right) - \frac{a}{n+1} \left( \frac{U^{n+1} - L^{n+1}}{U - L} \right). \]
For every \( a, U, \) and \( L \), there is a \( b \) that works.

5. Can functions other than polynomials be non-fundamental on some interval? If so, give some examples.

The obvious choices are exponentials and sines and cosines. Karl Gross, o Bellville High School contributed the following example:

Consider \( f(x) = \ln(1 + x) + c \) on \([0, 1]\). Then \( 2 \ln 2 - 1 + c = \ln 2 - \ln 1 \), so \( c = 1 - \ln 2 = \ln \left(\frac{e}{2}\right) \).

So, \( f(x) = \ln(1 + x) + \ln \left(\frac{e}{2}\right) \) is non-fundamental on \([0, 1]\). But we know that the Maclaurin
expansion for \( f(x) = \ln(1 + x) \) is

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{x^n}{n} + \ldots \quad \text{with} \quad a_0 = 0, \quad a_1 = 1, \quad a_2 = -\frac{1}{2}, \quad a_3 = \frac{1}{3}, \ldots
\]

We have from our earlier work that

\[
a_0 = \frac{a_1}{2} + \frac{2a_2}{3} + \frac{3a_3}{4} + \frac{4a_4}{5} + \frac{5a_5}{6} + \frac{6a_6}{7}
\]

makes the function non-fundamental, so we must have

\[
\ln \left( \frac{e}{2} \right) = \frac{1}{2}(1) + \frac{2}{3}\left( -\frac{1}{2} \right) + \frac{3}{4}\left( -\frac{1}{3} \right) + \ldots + \left( \frac{n}{n+1} \right) \left( \frac{(-1)^{n+1}}{n} \right) + \ldots
\]

and we can use the non-fundamental property to find the sum of the infinite series.