Calculus Challenge Problem #7

Due February 10, 2010

Non-Fundamental Functions: When Wrong is Right!

A student was wondering why I took off points for his solution to the following integration problem.

$$\int_0^1 x^2 + 2x + \tfrac{5}{3}\, dx = \left(1^2\right) + 2\left(1\right) + \tfrac{5}{3} - \left(\left(0^2\right) + 2\left(0\right) + \tfrac{5}{3}\right) = 3.$$

First, note the error in the student's work. I patiently showed him that the correct method was

$$\int_0^1 x^2 + 2x + \tfrac{5}{3}\, dx = \left(\frac{x^3}{3} + x^2 + \tfrac{5}{3}x\right)\Bigg|_0^1 = \left(\frac{1^3}{3}\right) + \left(1^2\right) + \tfrac{5}{3}\left(1\right) - \left(\left(\frac{0^3}{3}\right) + \left(0^2\right) + \tfrac{5}{3}\left(0\right)\right) = 3.$$

He, of course, argued that since his way was easier and got the same answer, it was a better solution. This got me wondering…

Are there other functions for which $\int_0^1 f\left(x\right)dx = f\left(1\right) - f\left(0\right)$? Since the fundamental theorem of calculus is usually required to evaluate a definite integral, we will call a function for which $\int_a^b f\left(x\right)dx = f\left(b\right) - f\left(a\right)$ a *non-fundamental* function on [a, b]. We see that $f\left(x\right) = x^2 + 2x + \tfrac{5}{3}$ is non-fundamental on [0, 1].

1. Find a linear function that is non-fundamental on [0, 1].

2. Are there any other quadratic functions that are non-fundamental on [0, 1]?

3. Can you find all polynomials that are non-fundamental on [0, 1]?

4. Must the interval of integration be [0, 1] for polynomials to be non-fundamental? Can we find a more general result?

5. Can functions other than polynomials be non-fundamental on some interval? If so, give some examples.

Note: The name, "non-fundamental function" is not a standard term. I just made it up to describe the property being considered.