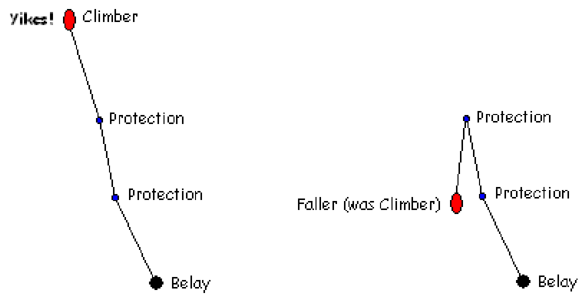
Calculus Challenge #8

Solutions due Wednesday, Feb. 24

Taking a Whipper



A rock climber periodically anchors clips in the rock through which his rope passes. In the event of a fall, this protection will limit the distance the climber falls. The rope is anchored by the belayer at the bottom, who is responsible for stopping the climbers fall.

At the bottom of the fall, the climber is whipped around as the rope stretches and reaches is maximum length to stop the fall. This is known as "taking a whipper". We will model the climbers fall and determine

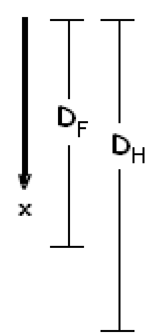
a) the velocity at the end of freefall.

b) the force on the climber due to the rope at the end of the fall. This given by $F = k\left(\dfrac{L^* - L}{L}\right)$, where L is the total length of the rope and L^* is its final stretched length.

c) a testing procedure to insure the rope can withstand a long fall and bring the climber to a safe stop.

The fall can be divided into two phases. During the initial part of the fall, the climber is in free-fall. Once the climber reaches the end of the rope, it begins to stretch, so Hooke's law now governs the descent.

1) First, the climber is in free fall for twice the distance separating the climber and the closest protection. Only gravity affects this component of the fall (we ignore the effect of air resistance and the wild gyrations of the falling climber's arms during the fall). The distance the climber free falls is denoted D_F. For this part of the fall, $F = ma = mg$. Let x represent the distance fallen at time t. Show that, if $a = g$, then the velocity of the climber (now faller) at $x = D_F$ is $v = \sqrt{2gD_F}$.



It is often helpful to be able to write equations that are given as functions of time *t* in terms of distance *x*. The easiest transformation uses the chain rule. So, $a = \frac{dv}{dt}$ and $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$. Since $\frac{dx}{dt} = v$, we have $\frac{dv}{dt} = v\frac{dv}{dx}$. This will be helpful in the next section.

Since $F = ma = mg$, we have $a = \frac{F}{m} = \frac{mg}{m} = g$. If $\frac{dv}{dt} = g$ then $v\frac{dv}{dx} = g$ so $\int v\frac{dv}{dx}dx = \int g\,dx$. Finally, we have $\frac{v^2}{2} = gx + c$. Since $v = 0$ when $x = 0$, we have $c = 0$ and $v = \sqrt{2gD_F}$.

2) Once the climber hits the end of the rope, it stretches. Climbing ropes are designed to stretch to minimize the force on the climber when the fall is finally arrested. The total distance of both components is denoted D_H. The subscript *H* is used to remind us that the last portion of the fall is subject to Hooke's Law. For this part of the fall, we can write $F = ma = mg - k\left(\frac{x - D_F}{L}\right)$ with $x > D_F$. Find $D_H - D_F$.

If $F = ma = mg - k\left(\frac{x - D_F}{L}\right)$, then $v\frac{dv}{dx} = g - \frac{k}{m}\left(\frac{x - D_F}{L}\right)$. Integrating, we find $\frac{v^2}{2} = gx - \frac{k}{2mL}\left(x - D_F\right)^2 + c$. Since $v = \sqrt{2gD_F}$ when $x = D_F$, we have

$\frac{2gD_F}{2} = gD_F + c$, so $c = 0$. So, $\frac{v^2}{2} = gx - \frac{k}{2mL}\left(x - D_F\right)^2$ and at $x = D_H$, we have $v = 0$.

Solving, we find that $D_H - D_F = \sqrt{\frac{2gmLD_H}{k}}$.

3) Find the force $F = k\left(\frac{L^* - L}{L}\right)$ on the climber at $x = D_H$ in terms of $\left(\frac{D_H}{L}\right)$, which is known as the fall ratio. Explain why $\left(\frac{D_H}{L}\right)$ has a maximum value of 2. Climbing rope is tested by dropping an 80 kg weight 5 meters. Explain why the equation above allows a 5 meter drop to test a rope that could easily be used to halt a 40 meter drop?

We have $F = k\left(\frac{L^* - L}{L}\right)$ and $D_H - D_F = \sqrt{\frac{2gmLD_H}{k}}$, so

$$F = \frac{k}{L}\sqrt{\frac{2gmLD_H}{k}} = \sqrt{2gmk\left(\frac{D_H}{L}\right)}.$$

The largest possible fall is when no protection is placed, so $\frac{D_H}{L}$ is approximately 2 (a little more due to the stretch in the rope). If there are any protections in place, the fall ratio will be less than two. So, since it doesn't matter what the length of the fall, just the fall ratio, a 5 meter drop is sufficient.