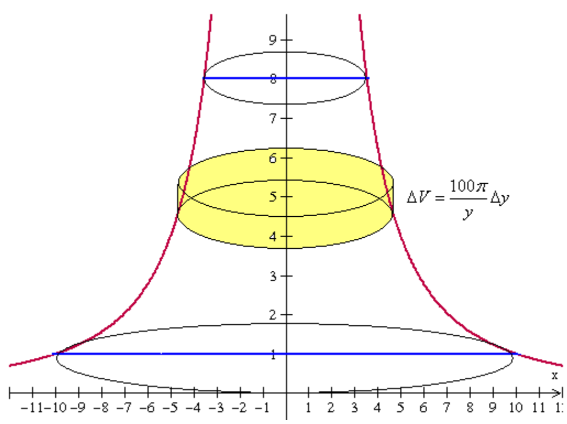
Calculus Challenge #9 **Solution**

Houdini's Escape

Houdini plans to have his feet shackled on the top of a concrete block which was placed on the bottom of a giant flask. The cross-sectional radius of the flask, measured in feet, is given as a function of the height *y* from the ground by the formula $r(y) = \frac{10}{\sqrt{y}}$, with the bottom of the flask at *y* = 1 foot. The flask is to be filled with water at a constant rate of 22π cubic feet per minute. Houdini's job is to escape the shackles before he drowns!

Houdini knows that he can escape the shackles in exactly 10 minutes. For dramatic effect, he wants to escape at the moment the water level reaches the top of his head. Houdini is 6 feet tall. In the design of the apparatus, Houdini can change only the height of the concrete block on which he stands.

1) Your first task is to find out how high this block should be. Express the volume of the water in the flask as a function of the height of the liquid above the ground level. Ignore the volume of the concrete block and of Houdini himself. What is the volume when the water level reaches the top of Houdini's head? What is the height of the block?



The volume of water in the flask is just a volume of revolution problem, with a volume element of $\Delta V = \frac{100\pi}{y}\Delta y$, so we need $\int_1^h \frac{100\pi}{y}dy = 100\pi\ln(h)$ cubic feet.. We need to pour water into the flask for 10 minutes at a rate of 22π cubic feet per minute, so a volume of 220π cubic feet.

To find *h*, we solve $220\pi = 100\pi\ln(h)$. Then $\ln(h) = 2.2$ and *h* = 9.025. We must subtract Houdini's height and the 1 foot for the bottom, so the block must be 2.205 feet in height.

2) Let *h*(*t*) be the height of the water above ground level at time t. In order to check the progress of his escape moment by moment, Houdini derives the equation for the rate of change $\frac{dh}{dt}$ as a function of *h*(*t*) itself. Find this equation. How fast is the water level changing when

the flask first starts to fill? How fast is the water level changing when the water just reaches the top of his head? Express $h(t)$ as a function of time.

Rewriting our equation from 1) in terms of time, we have $22\pi t = 100\pi \ln\left(h(t)\right)$, so $h(t) = e^{0.22t}$.

Then $\frac{dh}{dt} = .22h$. Initially, $h = 1$, so $\frac{dh}{dt} = .22$, so the water is rising slowly at 0.22 feet per minute.

When the depth is 9.025 feet, $\frac{dh}{dt} = .22(9.025) = 1.9855$, so the water is rising rapidly 1.9855 feet per minute.

3) Houdini would like to perform this trick with any flask. Generalize the result in b) by considering a flask with arbitrary cross-sectional radius $r(y)$ and constant inflow rate of $\frac{dV}{dt} = k$. Find $\frac{dh}{dt}$ as a function of $h(t)$.

In this case we have $\int_1^h \pi r(y)^2 dy = kt$. The integral is a function of h, so by the 2nd Fundamental Theorem of Calculus, $\frac{d}{dt}\int_1^h \pi r(y)^2 dy = \frac{d}{dt}kt$ and $\pi r\left(h(t)\right)^2 \frac{dh}{dt} = k$.

$$\frac{dh}{dt} = \frac{k}{\pi\left[r\left(h(t)\right)\right]^2}.$$

4) How would your calculations be altered if you take into account the volumes of Houdini, given Houdini's cross-sectional area is $A(y)$?

In this case, we need to remove the volume of Houdini. So, $\int_1^h \pi r(y)^2 dy - \int_{2.025}^h A(y)\,dy = kt$. At heights greater than 9.025, $A = 0$.

So $\frac{d}{dt}\int_1^h \pi r(y)^2 dy - \frac{d}{dt}\int_{2.025}^h A(y)\,dy = \frac{d}{dt}kt$ and $\pi\left(r(h)\right)^2 \frac{dh}{dt} - A(h)\frac{dh}{dt} = k$.

Finally, $$\frac{dh}{dt} = \frac{k}{\pi\left(r\left(h(t)\right)\right)^2 - A\left(h(t)\right)}.$$