

Calculus Challenge #12

SOLUTION

This is the last challenge until after the AP Exam

Arc Length Problems

You have probably noticed that there seem to be only a few functions whose arc lengths we can actually find. For example, we cannot find the length of an arc on the simplest functions like $y = x^2$, $y = \frac{1}{x}$, $y = e^x$, $y = \sin(x)$ since we cannot find an antiderivative for the integrand $\sqrt{1+(y')^2}$ (some of these integrals can be evaluated using advanced techniques that are not part of the AP curriculum). Your text probably has a few examples of “nice” functions that do work. Examples are $y = x^{3/2}$, $y = \frac{1}{8}x^2 - \ln(x)$, and $y = \frac{x^4 + 3}{6x}$. Clearly, these are contrived functions designed to work in the arc length formula. Where do these “nice” functions come from?

In this challenge, we will demonstrate one way to create functions that will “work” by insuring that $1+(y')^2$ is itself a perfect square. At least then you will have a fighting chance to evaluate the integral.

Let $y = f(x)$ be the function whose length we want to find. We want to insure that $1+(f'(x))^2 = g^2(x)$ for some integrable function g . This means that we require $g^2(x) - (f'(x))^2 = 1$.

1. If we define function h to be one of the factors of the difference of squares above, with $h(x) = g(x) + f'(x)$, we can show that $f'(x) = \frac{1}{2} \left(h(x) - \frac{1}{h(x)} \right)$. Verify that if $1+(f'(x))^2 = g^2(x)$ and $h(x) = g(x) + f'(x)$, then $f'(x) = \frac{1}{2} \left(h(x) - \frac{1}{h(x)} \right)$.

This is mostly algebra. If $1+(f'(x))^2 = g^2(x)$ then $(f')^2 - g^2 = 1$ (we suppress the function notation to make the terms a little easier to read) and $(f' - g)(f' + g) = 1$. We define

$h(x) = g(x) + f'(x)$ and by substitution, $(f' - g) \cdot h = 1$, so $f' = g - \frac{1}{h}$. Since $h = g + f'$, we have

$g = h - f'$ and $f' = h - f' - \frac{1}{h}$. Finally, $2f' = \left(h - \frac{1}{h} \right)$ or $f'(x) = \frac{1}{2} \left(h(x) - \frac{1}{h(x)} \right)$.

2. Now, pick nice functions for h . If $h(x) = x^{10}$, find a function f whose arch length we can find exactly on $[1, 2]$. Also find the length of the arc. Could you find the length on $[0, 1]$? Why or why not?

If $h(x) = x^{10}$, then $f'(x) = \frac{1}{2} \left(x^{10} - \frac{1}{x^{10}} \right)$. Note that we do have the perfect square we needed, since $1 + (f'(x))^2 = 1 + \frac{1}{4} \left(x^{20} - 2 + \frac{1}{x^{20}} \right) = \left(\frac{x^{10}}{2} + \frac{1}{2x^{10}} \right)^2$.

To find f , we integrate. $f(x) = \int f'(x) dx = \frac{1}{2} \int \left(x^{10} - \frac{1}{x^{10}} \right) dx = \frac{1}{2} \left(\frac{x^{11}}{11} + \frac{x^{-9}}{9} \right) = \frac{x^{11}}{22} + \frac{1}{18x^9} + c$.

The constant of integration doesn't affect the arc length, so we will let $c = 0$.

The arc length is

$$\int_1^2 \sqrt{1 + (f'(x))^2} dx = \int_1^2 \sqrt{\left(\frac{x^{10}}{2} + \frac{1}{2x^{10}} \right)^2} dx = \int_1^2 \left| \frac{x^{10}}{2} + \frac{1}{2x^{10}} \right| dx = \frac{2^{11}}{22} - \frac{1}{(18)2^9} + \frac{1}{99} = 93.1009.$$

You could not find the arc length on $[0, 1]$ since the improper integral diverges.

3. If $h(x) = x^n$, what is the function f whose length you want to find? Find the length of the arc of f on $[1, 2]$. What restrictions are there on n ?

Repeat problem 2 only with the general power $h(x) = x^n$.

If $h(x) = x^n$, then $f'(x) = \frac{1}{2} \left(x^n - \frac{1}{x^n} \right)$. Note that we do have the perfect square we needed, since $1 + (f'(x))^2 = 1 + \frac{1}{4} \left(x^{2n} - 2 + \frac{1}{x^{2n}} \right) = \left(\frac{x^n}{2} + \frac{1}{2x^n} \right)^2$.

To find f , we integrate. $f(x) = \int f'(x) dx = \frac{1}{2} \int \left(x^n - \frac{1}{x^n} \right) dx = \frac{1}{2} \left(\frac{x^{n+1}}{n+1} + \frac{x^{1-n}}{1-n} \right)$. Note that $n \neq 1, -1$.

The arc length is

$$\int_1^2 \sqrt{1+(f'(x))^2} dx = \int_1^2 \sqrt{\left(\frac{x^n}{2} + \frac{1}{2x^n}\right)^2} dx = \int_1^2 \left|\frac{x^n}{2} + \frac{1}{2x^n}\right| dx = \left(\frac{2^n}{n+1} - \frac{1}{2^{2-n}}\right) - \left(\frac{1}{2(n+1)} + \frac{1}{2(1-n)}\right).$$

4. Now let $h(x) = \tan(x)$.

If $h(x) = \tan x$, then $f'(x) = \frac{1}{2}(\tan x - \cot x)$. Note that we do have the perfect square we needed,

$$\text{since } 1+(f'(x))^2 = 1 + \frac{1}{4}(\tan^2 x - 2 + \cot^2 x) = \left(\frac{\tan x}{2} + \frac{\cot x}{2}\right)^2.$$

To find f , we integrate. $f(x) = \int f'(x) dx = \frac{1}{2} \int (\tan x - \cot x) dx$. By writing as sine and cosine, using u -substitution, we have $\frac{1}{2} \int (\tan x - \cot x) dx = -\frac{1}{2}(\ln(\cos x \sin x))$.

The arc length is

$$\int_1^2 \sqrt{1+(f'(x))^2} dx = \int_1^2 \sqrt{\left(\frac{x^{10}}{2} + \frac{1}{2x^{10}}\right)^2} dx = \int_1^2 \left|\frac{\tan x}{2} + \frac{\cot x}{2}\right| dx. \text{ Students should note that there is a discontinuity at } x = \frac{\pi}{2}, \text{ so this arc length cannot be found since the improper integral diverges.}$$

5. Find a function h that you think produces the most interesting result.

The simple function $h(x) = e^x$ is interesting, since the arc length problem generates the hyperbolic sine and cosine functions. If $h(x) = e^x$ then $f'(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$. Since $1 + \sinh^2(x) = \cosh^2(x)$, we have the necessary squared property.

To find f , we integrate. $f(x) = \int f'(x) dx = \frac{1}{2} \int (e^x - e^{-x}) dx = \frac{e^x + e^{-x}}{2} = \cosh(x)$.

Then the arc length is

$$\int_a^b \sqrt{1+(f'(x))^2} dx = \int_a^b |\cosh(x)| dx = |\sin(b)| - |\sin(a)|.$$