Calculus Challenge #12

Solutions due April 13, 2011

This is the last challenge until after the AP Exam

Arc Length Problems

You have probably noticed that there seem to be only a few functions whose arc lengths we can actually find. For example, we cannot find the length of an arc on the simplest functions like $y = x^2$, $y = \frac{1}{x}$, $y = e^x$, $y = \sin(x)$ since we cannot find an antiderivative for the integrand $\sqrt{1+(y')^2}$ (some of these integrals can be evaluated using advanced techniques that are not part of the AP curriculum). Your text probably has a few examples of "nice" functions that do work. Examples are $y = x^{3/2}$, $y = \frac{1}{8}x^2 - \ln(x)$, and $y = \frac{x^4+3}{6x}$. Clearly, these are contrived functions designed to work in the arc length formula. Where do these "nice" functions come from?

In this challenge, we will demonstrate one way to create functions that will "work" by insuring that $1+(y')^2$ is itself a perfect square. At least then you will have a fighting chance to evaluate the integral.

Let $y = f(x)$ be the function whose length we want to find. We want to insure that $1+(f'(x))^2 = g^2(x)$ for some integrable function *g*. This means that we require $g^2(x)-(f'(x))^2 = 1$.

1. If we define function h to be one of the factors of the difference of squares above, with $h(x) = g(x) + f'(x)$, we can show that $f'(x) = \frac{1}{2}\left(h(x) - \frac{1}{h(x)}\right)$. Verify that if $1+(f'(x))^2 = g^2(x)$ and $h(x) = g(x) + f'(x)$, then $f'(x) = \frac{1}{2}\left(h(x) - \frac{1}{h(x)}\right)$.

2. Now, pick nice functions for *h*. If $h(x) = x^{10}$, find a function f whose arch length we can find exactly on $[1,2]$. Also find the length of the arc. Could you find the length on $[0,1]$? Why or why not?

3. If $h(x) = x^n$, what is the function f whose length you want to find? Find the length of the arc of f on $[1,2]$. What restrictions are there on *n*?

4. Now let $h(x) = \tan(x)$.

5. Find a function *h* that you think produces the most interesting result.